Chapter 7 - Section A

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Contents

Exercises

Ex. 11

($\rightarrow$) Observe $v = v_U + v_{U^\perp}$ and $w = w_U + w_{U^\perp}$. Clearly,

$$\begin{aligned}
\langle Pv, w\rangle &= \langle v_U, w\rangle \\
&= \langle v_U, w_{U^\perp}\rangle + \langle v_U, w_U\rangle \\
&= 0 + \langle v_U, w_U\rangle \\
\langle v, Pw\rangle &= \langle v_{U^\perp}, w_U\rangle + \langle v_U, w_U\rangle \\
&= 0 + \langle v_U, w_U\rangle
\end{aligned}$$

($\leftarrow$) For $U = range\ T$ and $v = v_U + v_{U^\perp}$, we show $Tv = v_U$.

Lemma. $Tv_U = v_U$.

Since $v_U \in range\ T$, by definition we know $Tv_0 = v_U$. So $T(Tv_0) = Tv_0$ as $T^2 = T$, which concludes $Tv_U = v_U$.

Lemma. $Tv_{U^\perp} = 0$.

By definition we know $v_{U^\perp} \in (range\ T)^\perp$. But given T is *self-adjoint*, $(range\ T)^\perp = null\ T$. So $v_{U^\perp} \in null\ T$.

In conclusion, $Tv = Tv_U + Tv_{U^\perp} = v_U + 0 = v_U$.

Ex. 17

Fact. For *normal* T, $range\ T = range\ T^*$ and $null\ T = null\ T^*$. For any T, $range\ T = (null\ T^*)^\perp$. See *ex.16*.

Lemma. For *normal* T, $range\ T \cap null\ T = \{0\}$.

Observe $L.H.S = (null\ T^*)^\perp \cap (null\ T^*)$ by the aforementioned facts.

Theorem. $null\ T^k = null\ T$.

Clearly $null\ T \subset null\ T^k$ as $T0 = 0$ for any operator T. It remains to show $null\ T^k \subset null\ T$.

$$v \rightarrow^T v_1 \rightarrow^T v_2 \rightarrow^T \cdots \rightarrow^T v_k = 0.$$

$$v_{k-1} \in range\ T \cap null\ T, \text{ so } v_{k-1} = 0.$$

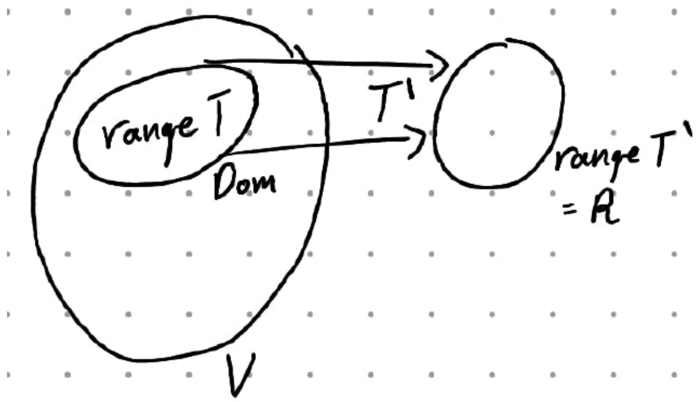
$$\cdots$$

$$v_1 \in rangeT \cap nullT, \text{ so } v_1 = 0.$$

Thus $Tv = v_1 = 0$, and $v \in null\ T$.

Theorem. *range* $T^k =$ *range* T.

Let T' be the same as T but restricted on subspace *range* T. Observe it is a linear operator.



We prove *null* $T' = \{0\}$. Observe for $v \in null\ T'$, $v \in range\ T \cap null\ T$, and hence $v = 0$. Clearly $T'0 = 0$ as $T0 = 0$ for any operator T.

It follows *dim null* $T' = 0$. By *The Fundamental Theorem of Linear Maps* (See Axler page 63), *dim range* $T =$ *dim range* T'. But by definition *range* $T' \subset$ *range* T, and therefore *range* $T' =$ *range* T.

We conclude $T[range\ T] = range\ T$, The image of *range* T under T is exactly *range* T. Clearly it suffices to prove our intended theorem.

Ex. 19

By normality we know *null* $T = (range\ T)^{\perp}$. So $(z_1, z_2, z_3) \perp v$, for any $v \in ran\ T$. It follows

$$\begin{aligned}(z_1, z_2, z_3) \cdot v &= 0\\ (z_1, z_2, z_3) \cdot T(1,1,1) &= 0\\ &= (z_1, z_2, z_3) \cdot (2,2,2) = 2z_1 + 2z_2 + 2z_3 = 2(z_1 + z_2 + z_3)\end{aligned}$$

Thus $z_1 + z_2 + z_3 = 0$.