Ch.10, Sec.01 - Bartle & Sherbert. Real Analysis

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January 10, 2024

Contents

Problems

1

a

By definition of a gauge, we have

$$\begin{aligned} t_i - \delta(t_i) &\leq x_{i-1} \\ x_i &\leq t_i + \delta(t_i) \end{aligned}$$

Implying,

$$\begin{aligned} x_i - x_{i-1} &\leq t_i + \delta(t_i) - x_{i-1} \\ -t_i + \delta(t_i) &\geq -x_{i-1} \end{aligned}$$

Concluding for all $i \in \{1, 2, \ldots, n\}$,

$$\begin{aligned} x_i - x_{i-1} &\leq t_i + \delta(t_i) - t_i + \delta(t_i) \\ &\leq 2\delta(t_i) \end{aligned}$$

b

Clearly $x_i - x_{i-1} \leq 2\delta^*$ for all $i \in \{1, 2, \ldots, n\}$. Then $\max\{x_i - x_{i-1}\} = ||\dot{p}|| \leq 2\delta^*$.

c

$\max\{x_i - x_{i-1}\} \leq \delta_* = inf\{\delta(t)\}$. Then $x_i - x_{i-1} \leq \delta_*$

$$\begin{aligned} x_i &\leq \delta(t_i) + x_{i-1} \\ &\leq \delta(t_i) + t_i \qquad \text{by def } x_{i-1} \leq t_i \end{aligned}$$

Analogously,

$$\begin{aligned} x_{i-1} &\geq -\delta_*(t_i) + x_i \\ &\geq -\delta_*(t_i) + t_i \qquad \text{by def } x_i \geq t_i \end{aligned}$$

Therefore, $[x_{i-1}, x_i] \subset [t_i - \delta(t_i), t_i + \delta(t_i)]$, i.e Q is $\delta-$fine.

d

2

a

Observe for interval $[x_{i-1}, x_i]$ for any partition,

$$[x_{i-1}, x_i] \cap [x_{j-1}, x_j] = \begin{cases} [x_{i-1}, x_i] & i = j \\ \{x_i\} & j = i+1 \\ \{x_{i-1}\} & j = i-1 \\ \phi & \text{otherwise} \end{cases}$$

It is easy to see considering any third interval containing a point x, necessarily implies two intervals share an intermediary point, violating the partitioning condition.

b

Yes. For example, on $[0, 1]$, we have the partition:

$$\begin{aligned} &([0, 1/4], 1/4), \\ &([1/4, 1/2], 1/4), \\ &([1/2, 3/4], 3/4), \\ &([3/4, 1], 3/4) \end{aligned}$$