Problem-Set 06

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Contents

Exercises

Ex. 1

skipped in hope of professionally read while solving the exercises, and well-gain from lectures.

Ex. 2

To definte shortest-path weight function δ, which satisfies the triangle inequality, enabling the second property of $\overline{w}$.

Ex. 3

For a cycle $c = v_0, v_1, \ldots, v_k = v_0$ we are given $w(c) = 0$. It is natural to ignore the case $k = 0$.

Recall the facts

1. $\overline{w}(u, v) \geq 0$
2. $\overline{w}(u, v) = w(u, v) + h(u) - h(v)$
3. $\overline{w}(p) = w(p) + h(v_0) - h(v_k)$ for path p

Lemma. 1 $\Sigma\overline{w}(v_i, v_{i+1}) = 0$

$$
\begin{aligned}
\overline{w}(c) &= w(c) + h(v_0) - h(v_k) \\
&= 0 + h(v_0) - h(v_0), \quad v_0 = v_k \\
&= 0
\end{aligned}
$$

If any $\overline{w}(v_i, v_{i+1}) > 0$ then $\overline{w}(c) > 0$, contradicting the proved above lemma.

Ex. 4

skipped in hope of professionally read while solving the exercises, and well-gain from lectures.

Ex. 5

In page 636 there is a hint of using *fibonacci-heabs*. I am not sure whether it is the key of solving the problem. Anyway, The exercise is postponed untill we gain a guidance from others. Skimming the chapter did not yield any promising clue to pursue.

Ex. 6

Same as *Ex. 5*

Problems

Prob. 1

a

Case $r = w_{i,j}$. Nothing to be done.

Case $r < w_{i,j}$. Check to see if new paths including edge (i, j) offer less-weight.

(For LaTeX issue we denote matrix Π by P)

```
for x = 0 to n
  for y = 0 to n
    if d_x,i + r + d_j,y < d_x,y
      d_x,y = d_x,i + r + d_j,y
      P(x,j) = i
      P(x,y) = P(j,y)
```

Observe $\Pi(x, i)$ is the same, and same for its recursive vertices. Similarly to $\Pi(j, y)$.

Complexity. $\mathcal{O}(V^2)$

Case $r > w_{i,j}$. For paths which do not depend on $w_{i,j}$, Nothing needs to be updated about them. If their paths are less or equal than any path which includes $w_{i,j}$ then obviously these paths are still optimal when the weight of $w_{i,j}$ increases. If for vertex x, $P(x, j)! = i$ then x shall never visit edge $\{i, j\}$.

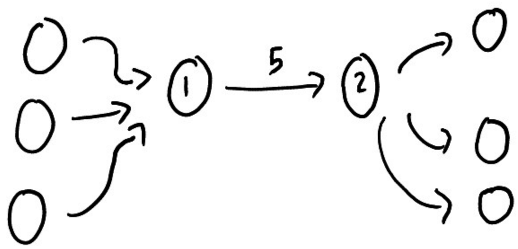
Our focus starts on vertices x whose $\Pi(x, j)$ equals i. For each such x and each arbitrary vertex y, We compute minimum paths from x to y and update if needed. Let D' and Π' denote minimum distance and predecessor matrices after updating the weight of edge $\{i, j\}$ to r, respectively. Any path $x \to y$ either consists of a single edge $\{x, y\}$ or contains an intermediate vertex between x and y. We loop on all vertices z to compute $D(x, z) + D(z, y)$ and then set $D'(x, y)$. However, we must check whether edge $\{i, j\}$ falls into the path $x \to z$ or $z \to y$. if NO, then we know $D'(x, z) = D(x, z)$ and $D'(z, y) = D(z, y)$. If YES, then the new weight of path $x \to y$ which equals $D(x, y) + (r - w_{i,j})$, is equal or less than the new weight $D(x, z) + D(z, y) + (r - w_{i,j})$. That follows by $D(x, y) \leq D(x, z) + D(z, y)$ as the additional weight $r - w_{i,j}$ is added on both sides of the inequality. In this case we know z won't offer a less-weight path. So we can restrict our focus on vertices z whose corresponding paths do not include edge $\{i, j\}$.

(For LaTeX issues we denote matrix Π by P)

```
isEdgeInPath(edge {i,j}, path x -> y, predecessor P)
  if P(x,j) != i
    return False

  s = y
  while P(x,s) != x
    if P(x,s) == j
      return False

  return True

Main()
  for x = 0 to n
    if P(x,j) = i
      for y = 0 to n
        minDistance = min{ edge (x,y) if exists, D(x,y) + (r - w_i,j) }
        minVertex = NULL
        isDistanceUpdated = False
        for z = 0 to n
          if isEdgeInPath( {i,j}, x -> y, P) OR z = x OR z = y
            continue to next iteration of z

          zDistance = D(x,z) + D(z,y)
          if (distance < minDistance)
            minDistance = zDistance
            minVertex = z
            isDistanceUpdated = True

         if isDistanceUpdated
           P'(x,y) = P(z,y)
           P'(x,z) = P(x,z)
```

Complexity. $\mathcal{O}(V^3)$

b



$|V| = 8$

$(1, 2, 10) = (i, j, r)$

$\#\text{Paths-modified} = 3 \cdot 3 = 9 = |V| + 1 = \Omega(|V|)$

c

In the same mannger matrices M and Π are maintainces *distances* and *predecessors*, We maintain also matrix W for the number of edges corresponding to $d_{i,j}$ in M. The algorithm then checks W before updating a new solution whether its number of edges is at most h.

Complexity. The overhead is constant over the original algorithm. In terms of parameters and h is postponed.

d

The algorithm constructs a series of matrices $L^1, L^2, .., L^{n-1}$ where $L^m = \left(l_{ij}^m\right)$, indicating shortest-paths of edges length at most m. The adapted algorithm terminates on L^h and outputs it.

Complexity. At most the complexity of the original algorithm.

e

Prob. 2

a

We prove if there are two different minimum spanning trees, T_a and T_b, Then we can construct a minimum spanning tree T_c whose weight is less than either of them.

We define:

- $E_a = T_a(E)$

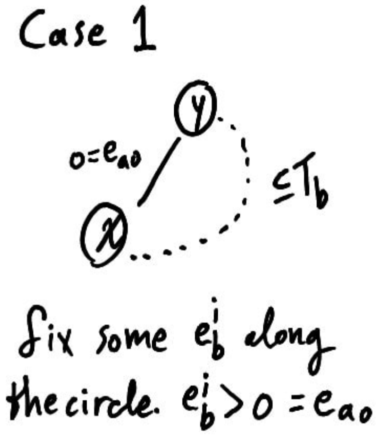
- $E_b = T_b(E)$
- $E_c = E_a \cap E_b$
- $E_{a-b} = E_a - E_b$
- $E_{b-a} = E_b - E_a$
- $E_{-c} = E_{a-b} \cup E_{b-a}$
- e_a, An edge in E_a
- e_{a0}, An edge in E_{a-b}

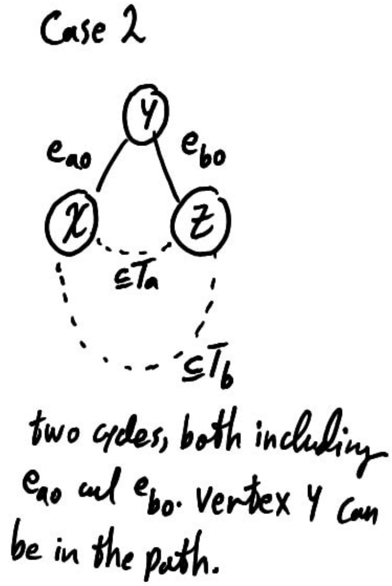
Lemma. 1 For an edge $e_{a0} = \{x, y\}$, x and y are connected by a path in T_b which does not include edge e_{a0}. Similarly for e_{b0}.
Follows immediately as by definition $e_{a0} \notin E_b$.

Lemma. 2 For an edge $e_{a0} = \{x, y\}$, There exists distinct edges e_b^1 and e_b^2 such that e_b^1 joins x and e_b^2 joins y in E_b. Similarly for e_{b0}.
Follows immediately by *Lemma 1.* Note the two edges e_b^1 and e_b^2 can share at most one vertix.

Lemma. 3 If there is a cycle where all edges are in E_a except exactly one edge e_b in E_b, and $w(e_b) < w(e_a^i)$ for some e_a^i in the cycle, then we can construct a MST $T_a' = T_a - e_a^i + e_b$ of weight less than T_a
Consider two vertices, v_1 and v_2, whose connectivity relies on edge $e_a^i = \{x, y\}$. The path is $p(v_1, x), (x, y), p(y, v_2)$. By adding e_b we know there is path $p_0(x, y) \neq (x, y)$, i.e x can reach y without edge (x, y). Therefore we can form an alternative path for v_1 and v_2 without relying on (x, y) by $p(v_1, x), p_0(x, y), p(y, v_2)$. Thus, Removing e_a^i is safe. Note It is clear neither $p(v_1, x)$ nor $p(v_2, x)$ contains edge (x, y) as that means there is an unnecessary cycle in the path.

Clearly E_{-c} is non-empty, Otherwise $T_a = T_b$. Without the loss of generality, Assume the selected element of E_{-c} is $\{x, z\} = e_{a0} \in E_{a-b}$. There are only two cases regarding the weight of e_{a0}.





Case 1: $w(e_{a0}) = 0$. By *Lemma 1* we know there is a path $p(x, y)$ which does not include e_{a0}. Clearly have a circle of, edges in E_b and exactly one edge in E_a. Since all weights of the graph are distinct and non-negative, $w(e_{a0})$ is strictly less than all edges in the circle. By *Lemma 3*, We can form a lower-weight MST. Contradiction.

Case 2: $w(e_{a0}) > 0$. By *Lemma 2* we get edges e_b^1 and e_b^2 in E_b where they contain vertices x and y. Clearly it is not possible for both e_b^1 and e_b^2 to be in E_a. Otherwise we would have a cycle in T_a contradicting the fact a tree has no cycles. It is easy to justify it by considering $T'_a = T_a - e_{a0}$. Without the loss of generality assume $e_b^1 \notin E_a$, i.e $e_b^1 = e_{b0}^1$. Denote e_{b0}^1 by $\{y, z\}$.

We claim there is a cycle of edges including e_{b0} and e_{a0}, Where all remaining edges are in E_a. By connectivity of T_a we know there is a path in T_a between x and z. Note the cycle is totally legit if it contained y. Similarly, There is a cycle of edges including e_{a0} and e_{b0}, Where all remainig edges are in E_b.

We know $e_{a0} \neq e_{b0}$. In either cases some edge is greater than the other. By *Lemma 3*, We get a lower-weight spanning tree. Contradiction.

b

Correctness. For any graph G, There is a unique sub-graph G_c, Such that for any cycle c in G whose all edges are in G_c except for exactly one edge e_x, The weight of e_x is the maximum along the whole cycle of c. The proof is nearly identical to *a*.

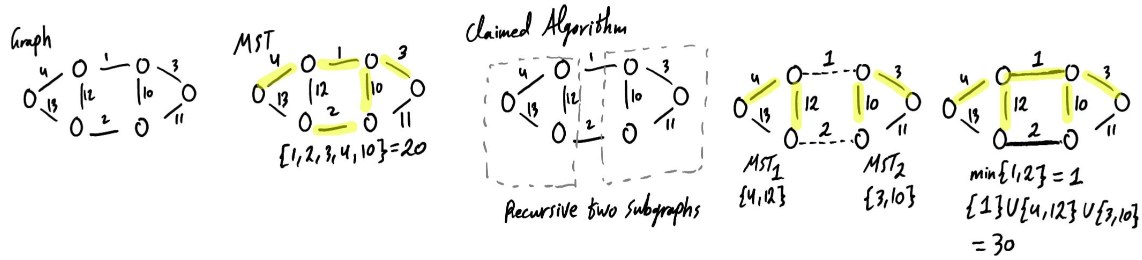
Clearly the MST exerts this property lest we construct another spanning-tree of less weight. Since the algorithm claimed here always prefers less-weight edges, It shall never contradict that property also. By uniqueness the claimed algorithm yields the MST.

Algorithm Description. "postponed"

Complexity Analysis. "postponed"

c

Counter-example:



d

Correctness. Yes. The proof is nearly identical to *a*.

Algorithm Description. "postponed"

Complexity Analysis. "postponed"

e

f