Problem-Set 07

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Contents

Exercises

Ex. 1
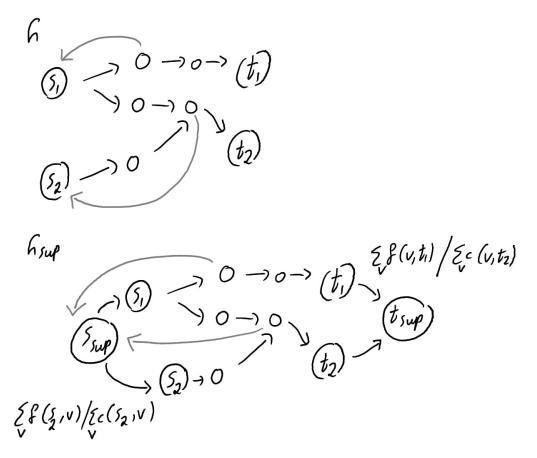
Will do while solving the exercises.

Ex. 2

Here are how definitions and properties, are adapted to the extension.

Definitions and Properties	Extended
Network	Instructed below
Source s	Sources $\{s_1, s_2, \ldots, s_{k_s}\}$
Sink t	Sinks $\{t_1, t_2, \ldots, t_{k_t}\}$
Capacity constraint	Same
Flow conservation	V - {sources} - {sinks}
flow value $|f|$	$\sum_{s_i}(\sum_v f(s_i, v) - \sum_v f(v, s_i))$

For brevity we illustrate the construction definition by the following diagram.

Observe $\sum_v f(s_{sup}, v) = \sum_{s_i} \sum_v f(s_i, v)$ and $\sum_v f(v, s_{sup}) = \sum_{s_i} \sum_v f(v, s_i)$ which

2

concludes our goal.

Ex. 3

Denote $f = \alpha f_1 + (1 - \alpha)f_2$. We show f satisfies both definitions of (1) *capacity* and (2) *reservation*.

(1) Clearly

$$0 \leq \alpha f_1(u, v) \leq \alpha c(u, v)$$
$$0 \leq (1 - \alpha)f_2(u, v) \leq (1 - \alpha)c(u, v)$$

Which concludes

$$\alpha f_1(u, v) + (1 - \alpha)f_2(u, v) \leq \alpha c(u, v) + (1 - \alpha)c(u, v) = c(u, v)$$

(2) Clearly,

$$\alpha \sum_v f_1(u, v) = \alpha \sum_v f_1(v, u)$$
$$(1 - \alpha) \sum_v f_2(u, v) = (1 - \alpha) \sum_v f_2(v, u)$$

Which concludes

$$\sum_v \alpha f_1(u, v) + (1 - \alpha)f_2(u, v) = \sum_v \alpha f_1(v, u) + (1 - \alpha)f_2(v, u)$$

Ex. 4

Intuitively and without proof, min-cut is $S = \{s, v_1, v_2, v_4\}$ and $T = \{v_3, t\}$.

c.

Ex. 5

Comment: I don't feel confident about the solution.

By *Ex. 2* we have a construction procedure from *multiple* to *single* with the same flow value. Intuitively and without proof, a flow of *multiple* corresponds to a flow of the same value in *single*. Then solving the *single* yields a max-flow for *multiple*.

Problems

Prob. 1

a

1

Notation. We say an edge (u, v) is saturated in F if $f(u, v) = c(u, v)$.

Fact. Since F is a max-flow, we know there is no augmenting path existing in it.

Lemma. Based on the fact, Lest a flow greater than $|f|$ gets constructed in F, The following cannot happen in F':

- Augmenting path, not containing edge (u, v).

- Augmenting path, containing (u, v), but (u, v) was not *saturated* in F.

- Augmenting path, containing (u, v), but $f(u, v)$ is decreased.

Lemma. Then, the possible cases of F' are:

- No augmenting path exists, Then $|f'| = |f|$.

- Augmenting path exists, containing (u, v), which was saturated in F, and flow increases it in F'. Since $c'(u, v) = f(u, v) + 1$, Then by definition of taking the minimum values along the path, It is concluded the increase in flow is at most 1.

If an augmenting path is taken in F', Then by the same line of reasoning, No second augmenting path is possible. Note $f'(u, v) = c'(u, v)$ now.

2
Similar.

A difference is the increase shall be at most k rather than at most 1, Following taking the minimum values along the augmenting path.

It is possible to have multiple augmenting paths, but even in that case, the overall increase shall be upperbounded by k.

The same line of reasoning leads to, any existing augmenting path, must be in the edge (u, v) whose capacity is increased.

3
Symmetric.

4
Similar.

b

Intuitively and without a proof, apply *Ford-Fulkerson* method, Searching for paths which only contains edge (u, v) whose capacity is increased. Intuitively that search restriction yields the upperbound $k \cdot (V + E)$.

c

Intuitively and without a proof, If the new capacity $c(u, v)$ is less than the flow of F, Fix the flow by decreasing flow values of edges which are in paths containing (u, v), Then similarly to *b*, apply *Ford-Fulkerson* but only on paths containing (u, v). Intuitively that search restriction yields the upperbound $k \cdot (V + E)$

Prob. 2

Out solution is based on intuition not a rigorous proof. Given a *directed graph* input, Construct a corresponding *flow network* where:

- A new source vertex s has a new edge with each s_i.
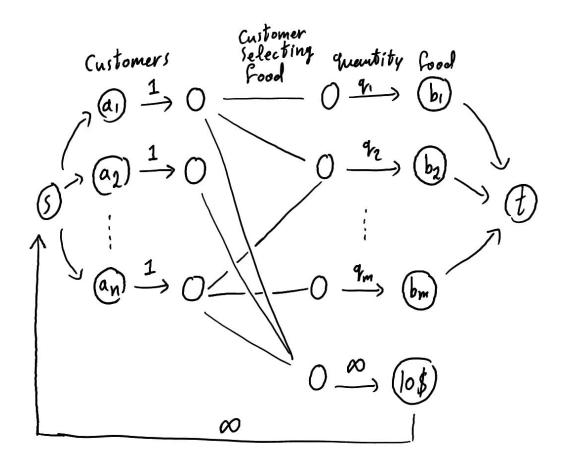- All edges have a capacity equal to 1.

Then we run *Ford-Fulkerson* to find the maximum flow. Now observe

- If max-flow $|f| = k$, output "Possible"
- if max-flow $|f| \neq k$, output "Impossible"

The goal of edge-disjoint paths is realized by setting all capacities to 1. If max-flow is k then each s_i has a path to t. If max-flow is not k then it must be less than k and means some s_i doesn't have a successful flow to t.

Prob. 3

Observe the following illustrative diagram which models the problem as a *max-flow* problem.

A customer a_i has an edge with food b_j in our constructed graph, If we are given $b_j \in A_i$. There is an additional edge for each customer to the special vertex *10$*.

Intuitively and without a proof, Maximizing a flow, corresponds to minimizing the 1-flow value output from each customer to the *10$* vertex as it does not reach vertex t and hence does not contribute to increasing the flow.