Problem-Set 08

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April 1, 2023

Contents

Exercises

Ex. 1

Done

Ex. 2

$$
\begin{aligned}
\text{Maximize } & d_y \\
\text{Subject to } & d_s = 0 \\
& d_t \leq d_s + w(s,t) \\
& d_y \leq d_s + w(s,y) \\
& d_y \leq d_t + w(t,y) \\
& d_x \leq d_t + w(t,x) \\
& d_t \leq d_y + w(y,t) \\
& d_x \leq d_y + w(y,x) \\
& d_z \leq d_y + w(y,z) \\
& d_z \leq d_x + w(x,z) \\
& d_x \leq d_z + w(z,x) \\
& d_s \leq d_z + w(z,s)
\end{aligned}
$$

Ex. 3

$$
\begin{aligned}
\text{Maximize } & \sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs} \\
\text{Subject to } f_{s,v_1} &\leq c(s, v_1) \\
f_{s,v_2} &\leq c(s, v_2) \\
f_{v_1,v_3} &\leq c(v_1, v_3) \\
f_{v_2,v_1} &\leq c(v_2, v_1) \\
f_{v_2,v_4} &\leq c(v_2, v_4) \\
f_{v_3,v_2} &\leq c(v_3, v_2) \\
f_{v_3,t} &\leq c(v_3, t) \\
f_{v_4,v_3} &\leq c(v_4, v_3) \\
f_{v_4,t} &\leq c(v_4, t)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{v \in V} f_{v_1 v} &= \sum_{v \in V} f_{v v_1} \\
\sum_{v \in V} f_{v_2 v} &= \sum_{v \in V} f_{v v_2} \\
\sum_{v \in V} f_{v_3 v} &= \sum_{v \in V} f_{v v_3} \\
\sum_{v \in V} f_{v_4 v} &= \sum_{v \in V} f_{v v_4}
\end{aligned}
$$

$$
\begin{aligned}
f_{sv_1} &\geq 0 \\
f_{sv_2} &\geq 0 \\
f_{v_1 v_3} &\geq 0 \\
f_{v_2 v_1} &\geq 0 \\
f_{v_2 v_4} &\geq 0 \\
f_{v_3 v_2} &\geq 0
\end{aligned}
$$

$$
f_{v_4 t} \geq 0
$$

Ex. 4

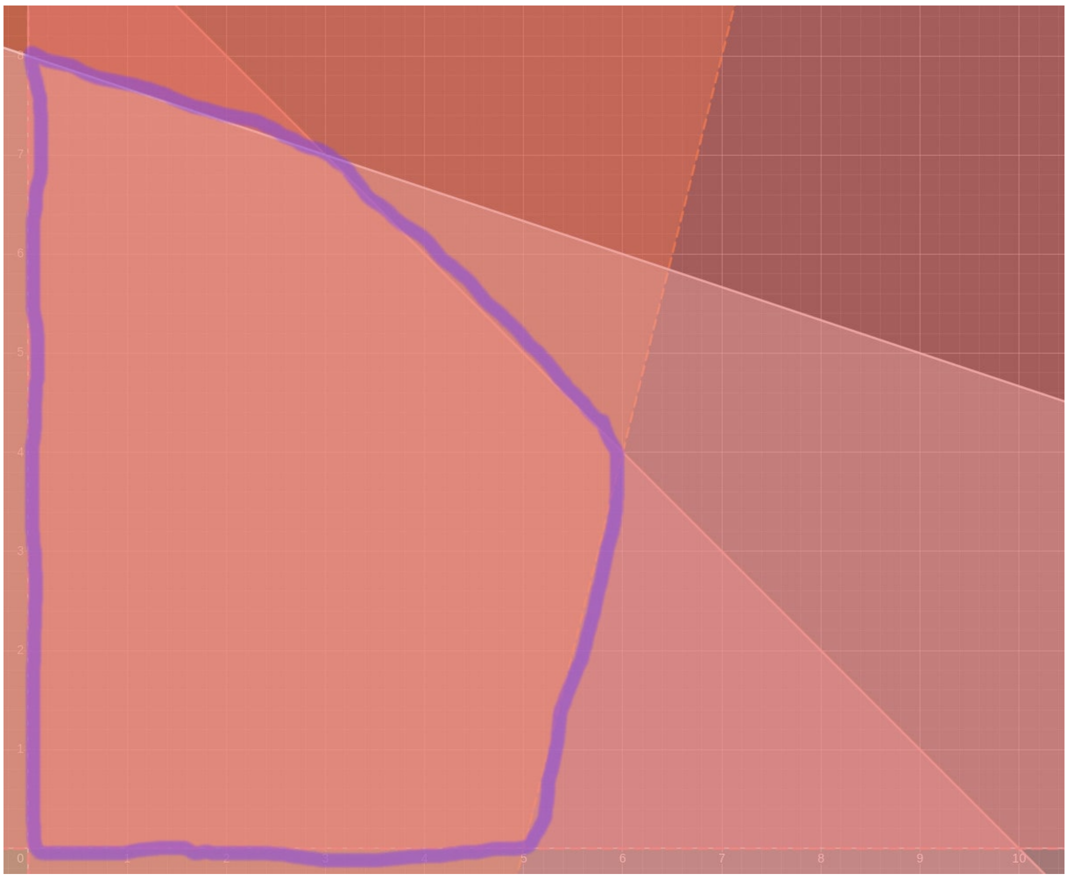
Ex. 5

Problems

Prob. 1

a

For the sake of time, We cheat by a drawn graph from desmos.



b

Standard Form

Here $n = 2$ and $m = 4$.

$$\begin{aligned} \text{Maximize } & 4(x_1) + 1(x_2) \\ \text{Subject to } & 1(x_1) + 1(x_2) \leq 10 \\ & 4(x_1) + -1(x_2) \leq 20 \\ & 1(x_1) + 3(x_2) \leq 24 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Slack Form

$$\begin{aligned} \text{Maximize } & 4(x_1) + 1(x_2) \\ \text{Subject to } & x_3 = -1(x_1) - 1(x_2) + 10 \\ & x_4 = -4(x_1) + 1(x_2) + 20 \\ & x_5 = -1(x_1) - 3(x_2) + 24 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

c

The solution form is $(x_1, x_2, x_3, x_4, x_5)$ where x_1, x_2 are basic and x_3, x_4, x_5 are non-basic.

Basic solution, by setting basic variables to zeros is $(0, 0, 10, 20, 24)$.

Interchange basic x_1 with non-basic x_4. Solving x_1 by the equation of x_4, We get $x_1 = 5 + \frac{1}{4}x_2 + \frac{-1}{4}x_4$. Substituting the new equation into remaining ones, We get:

$$\begin{aligned} \text{Maximize } & 20 + 2x_2 - x_4 \\ \text{Subject to } & x_1 = 5 + \frac{1}{4}x_2 + \frac{-1}{4}x_4 \\ & x_3 = 5 - \frac{5}{4}x_2 + \frac{1}{4}x_4 \\ & x_5 = 19 + \frac{-13}{4}x_2 + \frac{1}{4}x_4 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Solution $= (5, 0, 5, 4, 19)$ with objective equal to 20.

Interchange basic x_2 with non-basic x_3. Solving x_2 by the equation of x_3, We get

$x_2 = 4 - \frac{4}{5}x_3 + \frac{1}{5}x_4$. Substituting the new equation into remaining ones, We get:

$$\begin{aligned} \text{Maximize } & 28 - \frac{8}{5}x_3 - \frac{3}{5}x_4 \\ \text{Subject to } & x_1 = 6 - \frac{1}{5}x_3 - \frac{19}{20}x_4 \\ & x_2 = 4 - \frac{4}{5}x_3 + \frac{1}{5}x_4 \\ & x_5 = 6 + \frac{13}{5}x_3 - \frac{8}{20}x_4 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Solution $= (6, 4, 0, 0, 6)$ with objective equal to 28.

Quoting from CLRS, page 868: "At this point, all coefficients in the objective function are negative. As we shall see later in this chapter, this situation occurs only when we have rewritten the linear program so that the basic solution is an optimal solution."

d

$$\begin{aligned} \text{Minimize } & 10y_1 + 20y_2 + 24y_3 \\ \text{Subject to } & 1y_1 + 4y_2 + 1y_3 \geq 4 \\ & 1y_1 - 1y_2 + 3y_3 \geq 1 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

From c we know the optimal value is 28.

Prob. 2

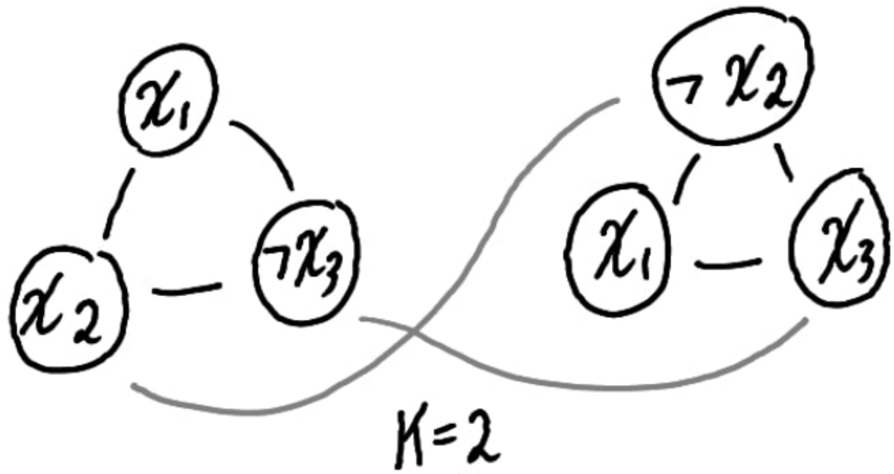
a

Given a $3SAT$ formula ϕ, Construct $\phi' = \phi \wedge (x_{n+1} \vee x_{n+2})$. Observe every *True* assignment of ϕ corresponds to three distinct *True* assignments of ϕ'. It follows ϕ is solvable if and only if ϕ' is, and the solution of $3SAT$ is basically the output on ϕ in formula of ϕ'. Therefore $3SAT$ is reduced to *TRIPLE-SAT*.

b

Observe the following illustrative example reducing $3SAT$ to $DONUT$.

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_1 \vee x_3)$$



- All profits of vertices equal exactly 1.
- Every clause is transformed to a complete 3-vertices sub-graph. It ensures Every 1 counted of k must be of a distinct sub-graph.
- k, the profit threshold to be satisfied, is equal to the number of clauses. It ensures every clause to be satisfied corresponds to counting 1 of k.
- For consistency, There is an edge between every v_i and $\neg v_i$, So that we cannot select both of them.
- It is not problemetic to select v_i multiple times. It is interpreted by many clauses being satisfied by the same literal.

Now it is clear, Given a problem instance x, The transformed construction $f(x)$, achieves x is $3SAT$ satisfiable if and only if $f(x)$ has a subset of profit at least k.

c

We reduce from *subset-sum* problem in CLRS p.1097.

34.5.5 The subset-sum problem

We next consider an arithmetic NP-complete problem. In the ***subset-sum problem***, we are given a finite set S of positive integers and an integer ***target*** $t > 0$. We ask whether there exists a subset $S' \subseteq S$ whose elements sum to t. For example, if $S = \{1, 2, 7, 14, 49, 98, 343, 686, 2409, 2793, 16808, 17206, 117705, 117993\}$ and $t = 138457$, then the subset $S' = \{1, 2, 7, 98, 343, 686, 2409, 17206, 117705\}$ is a solution.

As usual, we define the problem as a language:

$$\text{SUBSET-SUM} = \{\langle S, t \rangle : \text{there exists a subset } S' \subseteq S \text{ such that } t = \textstyle\sum_{s \in S'} s\} .$$

Given an arbitrary subset-sum set S, Transform each integer i to a task with time i and profit i. All deadlines are set to the given t of sum aimmed to find. Think of time as a discrete sequence of seconds.

Observe the maximum obtainable profit of machines is t. Observe also if that maximum is achieved then the integers corresponding to machines form a subset whose sum is t. On the other hand, If machines' profit found does not meet t then there is no subset of integers summing to t, As otherwise machines found won't be maximal.

The problem can be stated as a decision problem, by the existince of machines whose profit is at least k. The transformation between optimization and decision problems is easily done by a logarithmic binary search. For brevity we omit those details.