Problem-Set 09

Mostafa Touny

April 20, 2023

Contents

Exercises

Ex. 1

Ex. 2

Ex. 3

Ex. 4

Problems

Prob. 1

a

Consider

$$\begin{aligned} A &= \{1, 2\} \\ S &= \{1, 50\} \\ V &= \{2, 50\} \\ B &= 50 \end{aligned}$$

Observe the optimal solution is $C = 50$ while *Alg1* is of value $C* = 2$. Therefore the approximation ratio is $2/50 = 1/25$.

It is easy to see the number 50 can be set arbitrarily larger, and therefore we can reach the desired unbounded approximation ratio.

b

We follow the same convention of assuming the given indices order follow non-increasing order of their densities.

For the first index i such that $\sum_{j=1}^{i} v_j > B$, denote items up to $i-1$ by *Max-Dens-Items* and ith item by *First-Overweight*. For item a_i denote $D(a_i)$ to be the density of a_i, i.e the *value* per one unit of *weight*.

Take *D(First-Overweight)* and multiply it by the slack weight in B after consuming weights of *Max-Dens-Items*. Sum the resulting value along values of *Max-Dens-Items* and let $V_{maxDensities}$ denote that sum.

It is very clear $C* \leq V_{maxDensities}$ as we fully utilized the space of B with maximum possible densities.

Let's return to *Alg2* and note how similar it is to the way we defined $V_{maxDensities}$. Remark that $C = max\{V_{maxDensities}, \textit{value of First-Overweight}\}$. We have two cases:

- Case 1. Weights of *Max-Dens-Items* $\geq \frac{B}{2}$.
  Then $C \geq \frac{V_{maxDensities}}{2}$, As *Max-Dens-Items* accounts for more than 50% of $V_{maxDensities}$.
- Case 2. Weights of *Max-Dens-Items* $< \frac{B}{2}$.
  Then the weight of *First-Overweight* is greater than $\frac{B}{2}$. It follows $V_{maxDensities}$ is contributed only by *Max-Dens-Items* and *First-Overweight*. Observe one of them must contribute at least 50% of $V_{maxDensities}$. By definition, that one shall be selected by *Alg2*, and therefore $C \geq \frac{V_{maxDensities}}{2}$.

c

That is a standard dynamic programming problem whose solution can be found in any textbook. For brevity we only show the recurrece relation.

Base: $S_{1,v} = w(a_1)$ if $w(a_1) = v$.
Induction Step: $S_{i,v} = min\{S_{i-1,v}, w(a_i) + S_{i-1,v-v_i}\}$

d

Polynomial Time Complexity. Observe the time complexity of $Alg3$ is $\mathcal{O}(n^2V)$, As the memoization table is:

value\items	1	..	n
1			
..			
nV			

Since $Alg4$ basically runs $Alg3$ with additional linear operations, Its time complexity is $\mathcal{O}(n^2V')$, where V' is similarly defined but on scaled values v_i'.

Clearly $V' = \lfloor \frac{V}{V} \cdot \frac{n}{\epsilon} \rfloor = \lfloor \frac{n}{\epsilon} \rfloor$. So complexity of $Alg4$ can be re-written as $\mathcal{O}(n^3 \cdot \frac{1}{\epsilon})$.

Approximation Scheme. The idea is to use the bound of **b** but on scaled values v_i', then reverse the scaling to reach the intended ratio.

Let U be the upper-bound of optimal solutions which we defined earlier in **b** on given values v_i. Let U' be similarly defined but on scaled values v_i'. Define function f so that it scales value as mentioned by the author. Let C and C' denote the value of the subset solution obtained by $Alg4$ but on given and scaled values respectively.

From **b**, We know there exists a solution on scaled values v_i' whose approximation ratio is 2, out of U'. Then trivially the optimal solution also can deviate by a ratio of at most 2 out of U'.

Observe if we scaled back a value then the calculated value is no greater than the original given value, since we are taking ceils. In other words, $f^{-1}(v_i') \leq v_i$.

Joining all these remarks:

$$C' \geq \frac{1}{2}U'$$
$$C \geq f^{-1}(C') \geq \frac{1}{2}f^{-1}(U') = U$$

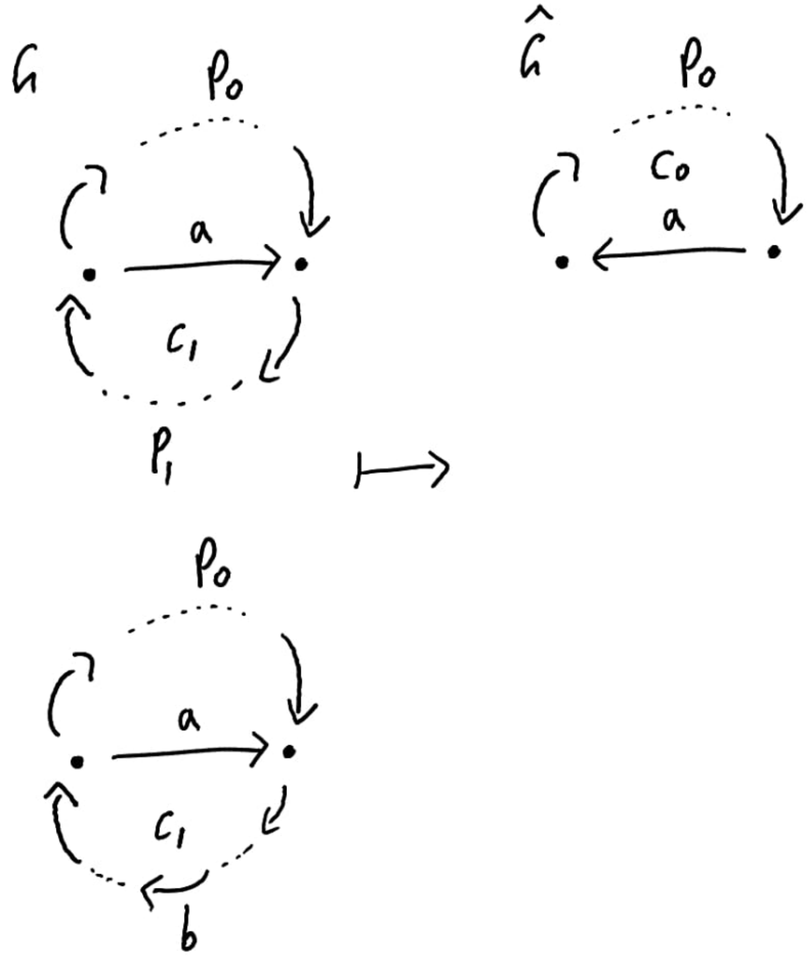
From **b**, That suffices to concluding $Alg4$ is an approximation scheme.

Prob. 2

a

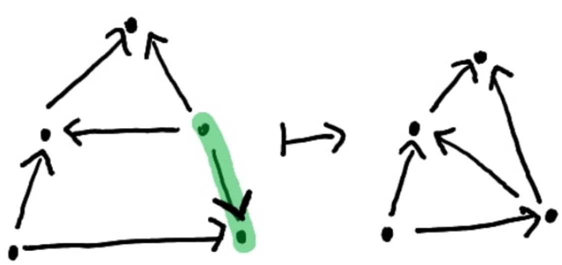
Assume for the sake of contradiction there is a cycle c_0 in the reversed graph $\hat{G}$. Then it must contain an edge from A. Otherwise c_0 would also be in graph G and by definition it must contain an edge from A. Call that edge a. Returning to G, a would be reversed as in the figure below. It is possible to have edges other than a in cycle c_0 which would also be reversed in graph G. In this case p_0 would be constructed by taking the corresponding sub-cycles into it.

Since A is minimal there must be a cycle c_1 in graph G which would not be covered if not for a. Observe we have cycle c_2 constructed by paths p_0 and p_1. What covers c_2 in G? Clearly no edge in path p_0 would do that since we already considered all edges of A we might encounter and took a sub-cycle avoiding them. Then c_2 is covered by edge b in path p_1 which is part of the cycle c_1. That contradicts c_1 being a cycle only covered by edge a. **QED**.



b

Remove all isolated vertices as they are irrelevant to cycles. Iteratively *contract* edges if they are not a side of a triangle as in the following figure.

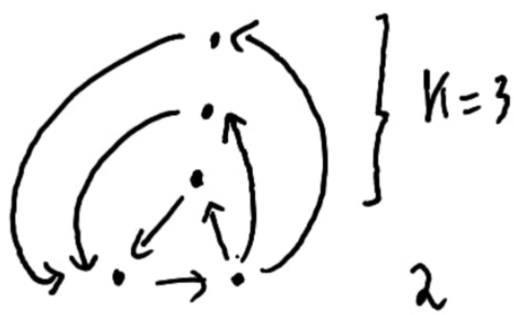


By definition, Each edge of the resulting graph is a side of a triangle. Observe the graph is still equivalent to the previous one, When it comes to cycles. Intuitively we just condensed the length of cycles.

For a single edge e, Consider the number of different triangles it is a side of. If the number is greater than k then we must have $e \in S$; Otherwise, To cover all of these triangles, We will need more than k edges. Note any two different triangles can share at most one edge. Remove edge e, and *contract* edges as needed if they are no longer a side of a triangle (suffices also to maintain no isolated vertices). Output the resulting graph as $\hat{G}$ but with a capacity of at most $k-1$ edges to cover all of its cycles.

After repeating this process, We will have a graph where each edge is a side of a triangles, whose count is no more than k. Also each vertex is part of a cycle. We show now the number of vertices is upper-bounded by $k^2 + 2k$. They key idea is, If there is an additional vertex, We will have cycles more than what k edges can accommodate.

For a single edge e, It can cover at most k cycles. Vertices in those cycles are exactly, 2 of the edge itself, and k for each cycle. That is a total of $2+k$. See the picture below:



Considering all edges of A, The total we get is $k(2+k) = 2k + k^2$.

It is clear now we cannot have vertices greater than that number. As by our graph structure that vertex v would be part of a cycle, and we have already consumed the maximum number of cycles k edges can cover. In other words, We will miss a cycle which contains vertex v.

c

It suffices to have a polynomial-time algorithm of the kernlization procedure we illustrated.

- Degrees of vertices are computed by a linear scan of edges, $\mathcal{O}(|E|)$.
- Contracting edges takes at most $\mathcal{O}(|E|^2)$.
- Computing number of triangles for each edge takes at most $\mathcal{O}(|E|(|E|+|V|))$ by a trivial graph search, made for each edge.
- Removing edges consumes $\mathcal{O}(|E|)$.

Since each step is polynomial in the size of the input, The sum of these sub-routines is polynomial also.