Harvey's Notes I - Chapter 02

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Contents

Exercises

Ex. 1

Part I

Take $m = n - 1$, and let R be the algorithm's output. $R = 0$ if and only if $RNG()$ returned 0 or $n - 1$. So $Pr[R = 0] = 2/n$ and $Pr[R = i] = 1/n$ for $i \neq 0$.

Part II

Define $k = \lfloor n/m \rfloor$, So k is the greatest integer such that $mk \leq n$. Define $r = n \mod mk$, so $n = mk + r$ where $0 \leq r < m$.

```
def goodSampler(m)
  k = floor( n/m )

  do
    r = RNG()
  while r >= mk

  return r mod m
```

Uniformity. Assume the algorithm terminates. So we are given $r < mk$ and we want to prove $Pr[r \mod m = i \mid r < mk] = \frac{1}{m}$ for $i \in [m]$. Observe

$$\begin{aligned} Pr[r \mod m = i \mid r < mk] &= \frac{Pr[r \mod m = i \cap r < mk]}{Pr[r < mk]} \\ &= \frac{k/n}{mk/n} = \frac{k}{n} \cdot \frac{n}{mk} = \frac{1}{m} \end{aligned}$$

Recall by uniformity the probability is basically the number of outcomes satisfying the event over all possible outcomes. Clearly the k outcomes of r yielding i by the algorithm are $(0)m + i, (1)m + i, (2)m + i, \ldots, (k-1)m + (i)$.

Time Complexity. For an iteration of `do-while`, probability of termination is mk/n. So in expectation it takes n/mk trials until it terminates. It follows

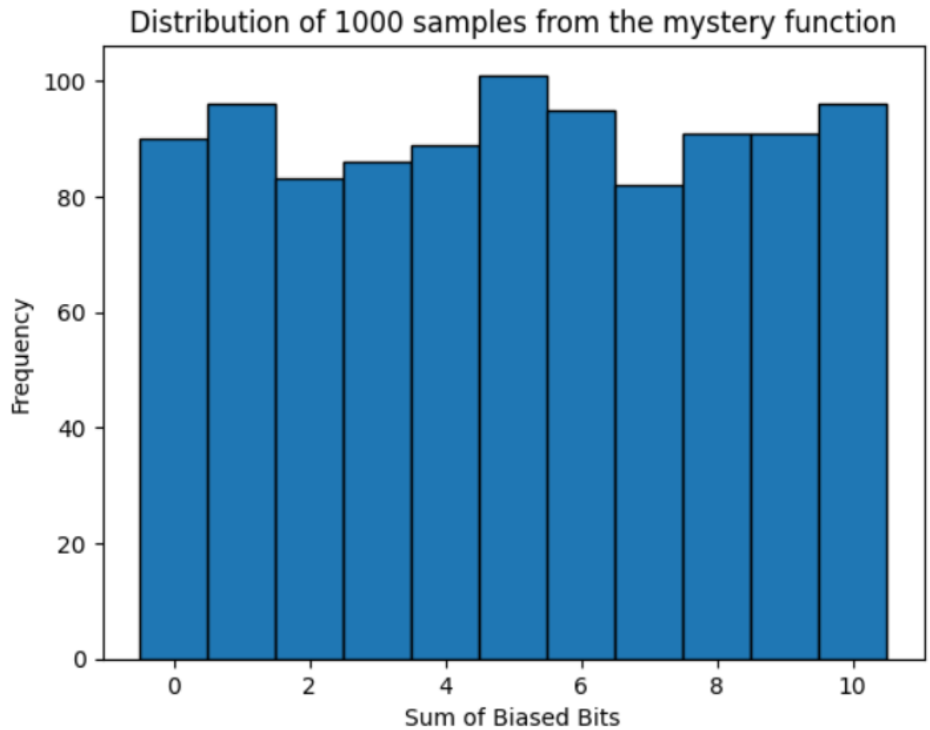
$$1 \leq \frac{n}{mk} = 1 + \frac{r}{mk} < 1 + \frac{m}{mk} = 1 + \frac{1}{k} \leq 2.$$

Concluding its time is $\mathcal{O}(1)$.

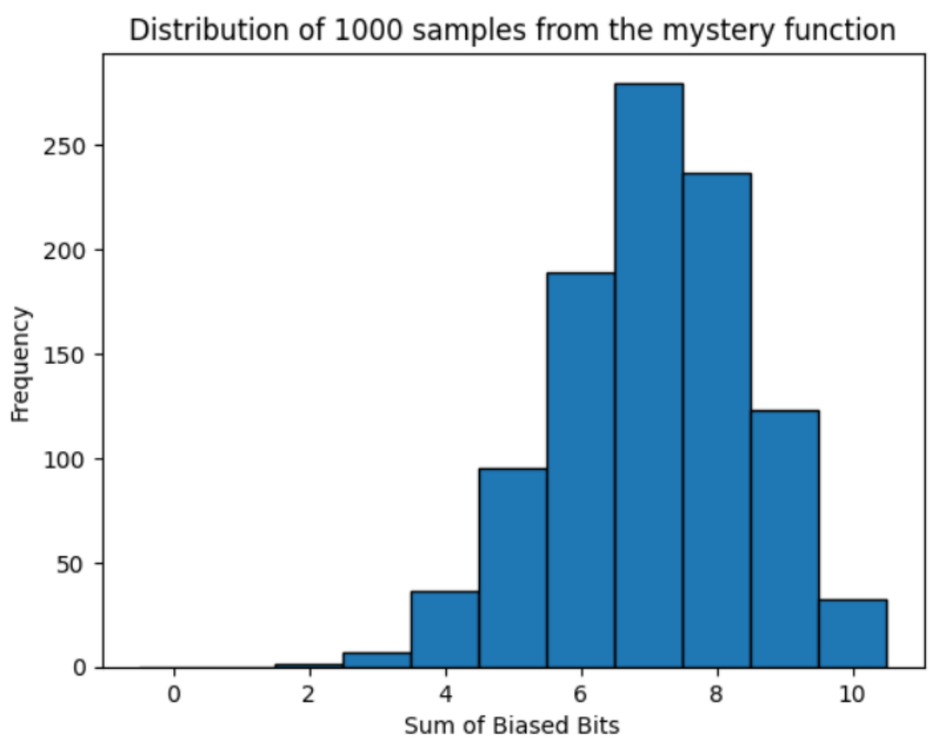
Ex. 2

Distribution. see *ex-2.2* notebook.

The distribution of the given psuedo-code seems uniform.



The distribution but with modifying the probability to be hardcoded `p = 0.7` rather than `p = ContinuousUniform()` in line 2, seems normal.



Recall we know in expectation we will get 7 1-bits out of 10 by linearity of random variables.

Remark. $X = \sum X_i^n$ is equivalent to number of 1s tossed.

Lemma. Probability of tossing k $1s$.

For some fixed probability of getting 1 p, and number of coin tosses n, The probability of drawing k $1s$ is $Pr[X = k] = (p)^k(1-p)^{n-k}\binom{n}{k}$, Since the distribution of coin tossing is binomial.

Ex. 3

Part I

Algorithm.

```
# input: Probabilities P[i]
# output: category sampled
def categoricalSampler( P[1..k] )

  # initially the universe is all probabilities
  totalProb = sum( P[1..k] )

  # for each ith probability
  for i in 1..k

    # compute P[i] probability in ratio to the universe
    i_prob_uni = (1/totalProb) * P[i]

    # return i by that probability
    if biasedBit( i_prob_uni )
      return i

    # remove P[i] from the universe
    totalProb -= P[i]
```

Correctness. Computing a probability out of a subset of probabilities.

We want to compute a probability but in ratio to some subset of probabilities.

For example if $Pr[X = i] = 1/4$ for $i \in \{1,2,3,4\}$, But we are given $X \notin \{1,2\}$. Then $Pr[X = 3 \mid X \notin \{1,2\}] = \frac{Pr[X = 3 \cap X \notin \{1,2\}]}{Pr[X \notin \{1,2\}]} = \frac{1/4}{1/2} = 2 \cdot \frac{1}{4}$.

Generally we want to find x where, For sum S of some subset of probabilities, $\frac{Pr[X=i]}{S} = \frac{x}{1}$, so $x = \frac{1}{S} \cdot Pr[X = i]$.

Correctness. `categoricalSampler` returns a category.

If the algorithm reached iteration k, `i_prob_uni` would be 1 so `biasedBit` surely fires.

Time Complexity. Clearly $\mathcal{O}(k)$.

Part II

Algorithm

```
# input: probabilities P[i]
# output: cumulative sum of probabilities S[i]
def cumulativeSum( P[1..k] )

    # S[i] is sum up to P[i]
    S = []

    sum = 0

    # compute & append S[i]
    for i in 1..k

        # cumulative sum up to P[i]
        sum += P[i]

        # append as S[i]
        S.append( sum )

    return S

# input: probabilities P[i], and cumulative probabilities S[i]
# output: sampled category i
def recursiveSampler( P[l..r], S[l..r] )
    # base case. universe is one category so its probability is 1
    if l = r
        return l

    # center index
    mid = floor( (r-l)/2 )

    # total probability of P[l..r]
    totalProb = S[r] - S[l] + P[l]

    # probability of cumulative half of P in ratio to the universe
    prob_uni = (1/totalProb) * S[mid]

    # toss a coin by cumulative probability of half of P
    if biasedBit( prob_uni )

        # if True then the sample is restricted to them
        return recursiveSampler( P[l..mid], S[l..mid] )
```

```
    else

        # if False then the sample is not any of them
        return recursiveSampler( P[mid+1..r], S[mid+1..r] )

# input: probabilities P[i]
# output: sampled category i
def categoricalSampler2( P[1..k] )

    # preprocessing, computing cumulative sum of probabilities
    S = cumulativeSum( P[1..k] )

    # sample a category
    return recursiveSampler( P, S )
```

Correctness. `recursiveSampler` won't ever reach array `P` of size zero.

That can only happen if either `mid = r` or `mid = l`, but then `S[mid] = 1` or `S[mid] = 0` respectively. Contradiction.

Correctness. The algorithm samples category i with probability $P[i]$.

The remarks from *Part I* holds here. We show a more formal proof.

Let $X = z$ denote the event of sampling category z. Let $j_1, j_2, \dots, j_{k-1}$ be the remaining categories. Then $Pr[X = z] = Pr[X \neq j_1 \cap X \neq j_2 \cap \dots \cap X \neq j_{k-1}]$. Partition js on subsets of outcomes $O_1, O_2, \dots, O_{\log k}$.

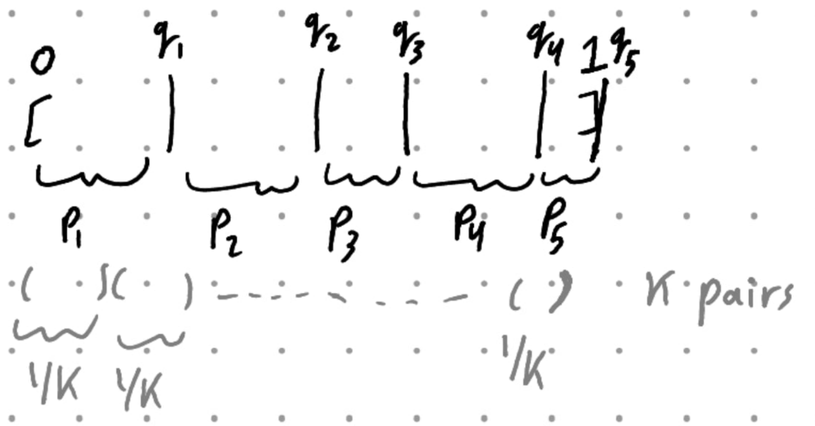
$$\begin{aligned} Pr[X = z] &= Pr[X \notin O_1 \cap X \notin O_2 \cap \dots \cap X \notin O_{\log k}] \\ &= Pr[X \notin O_1 \mid X \notin O_2 \cap \dots \cap X \notin O_{\log k}] \\ &\cdot Pr[X \notin O_2 \mid X \notin O_3 \dots \cap X \notin O_{\log k}] \cdot .. \cdot Pr[X \notin O_{\log k}] \end{aligned}$$

Let R be the algorithm's output, and let R_i correspond to `biasedBit` in iteration i. Clearly $Pr[R \in \{l, l+1, \dots, mid\}] = Pr[R_i = True]$ and $Pr[R \in \{mid+1, mid+2, \dots, r\}] = Pr[R_i = False]$. The algorithm samples z if and only if $R_1 = x_1 \cap R_2 = x_2 \cap \dots \cap R_{\log k} = x_{\log k}$ corresponding to $x_i = True$ if $z \notin \{mid + 1, \dots, r\} = O_i$. In other words the algorithm satisfies the definition of $Pr[X = z]$.

Time Complexity. Clearly `cumulativeSum` takes $\mathcal{O}(k)$ and `recursiveSampler` takes $\mathcal{O}(\log k)$

Ex. 4

Idea. See the following sketch for an intuition.



- Compute cumulative sums q_i.
- Construct k intervals, each of size $1/k$.
- If some q_i is sandwiched in some interval, separate that interval to two pieces.
- Accordingly label pieces which p_i they belong to.
- Sample a `UniformLBitInteger` to select a uniformly random interval out of those k intervals.
- If it is separated, Toss a coin by `BiasedBit` to decide a piece.
- Output the piece's label.

Algorithm.

```
# input: probabilities P[i]
# output: cumulative sum of probabilities S[i]
def cumulativeSum( P[1..k] )
    S = []
    sum = 0

    for i in 1..k
        sum += P[i]
        S.append(sum)

    return S


# input: integer k
# output: k-intervals of size 1/k each
def kIntervalsConstruction( k )
```

```
    # start, end, and separation points, decides a pair of intervals
    # end - start = 1/k
    intervals = [
        [st, end, sep, label]
    ]

    # construct the k intervals
    for i in 1..k

        # length of interval is 1/k
        intervals.append(
            [ (i-1)/k, i/k, NULL, NULL ]
        )

    return intervals


# input: 2d-array of k-intervals, and the cumulative sum S
# output: None. It modifies the 2d-array to be filled with separators and labels
def sepPointsFromArray( Inter[1..k, 4], S[1..k] )

    # pointer for S
    S_poi = 1

    # for each kth interval
    for i in 1..k

        # if S[S_poi] is contained in the kth interval
        if Inter[i][st] <= S[ S_poi ] <= Inter[i][en]

            # set S[S_poi] as the separator
            Inter[i][sep] = S[ S_poi ]

            # consider next index
            S_poi += 1

        # otherwise, leave the separator with NULL

        # in either cases cache S_poi
        # latter case. whole interval is labeled by S_poi
        # former case. first piece is labeled by S_poi and
        #   second piece is labeled by S_poi+1
        Inter[i][label] = S_poi


# input: k-probabilities P[i]
```

```
# output: k intervals, each of size 1/k, possibly separated to have pairs
def intervalToPieces( P[1..k] )

    # S[i] is cumulative sum up to P[i]
    S = cumulativeSum( P )

    # construct k-intervals of size 1/k each
    intervals = kIntervalsConstruction( k )

    # in-place fill "sep" and "label" cells in intervals
    sepPointsFromArray(intervals, S)

    return intervals


# input: Probabilities P[i]
# output: sampled category i
def sample( P[1..k] )

    # preprocessing is not needed to be called for every sample call
    intervals = intervalToPieces( P )

    # uniform random index of intervals
    randIndex = uniformLBitInteger(lg k)

    # uniform random interval from intervals
    randInterval = intervals[ randIndex ]

    # if it has no separator
    if randInterval[sep] = NULL

        # then it belongs to cached label
        return randInterval[label]
    else

    # it has a separator

        # compute proportion of separator in ratio to the interval of size 1/k
        prop = (randInterval[sep] - randInterval[st]) / (1/k)

        # toss a coin by a probability proportional to the separator
        if biasedBit(prop)

            # if it lands before the separator then we are in a piece labeled by "cached lab
            return randInterval[label]
        else
```

```
        # if it lands after the separator then we are in the piece next to one labeled b
        return randInterval[label] + 1
```

Correctness. No probabilistic claim is outside what we proved and illustrated in previous exercises.

Time Complexity. Preprocessing consumes $O(k)$, Since all its subroutines take $O(k)$ each. Sampling consumes $O(1)$ since both `uniformLBitInteger` and `biasedBit` consume $O(1)$.

Ex. 5

Postponed.

Ex. 6

***Note.** It feels weird we derived a solution better than the requested bound.

Algorithm

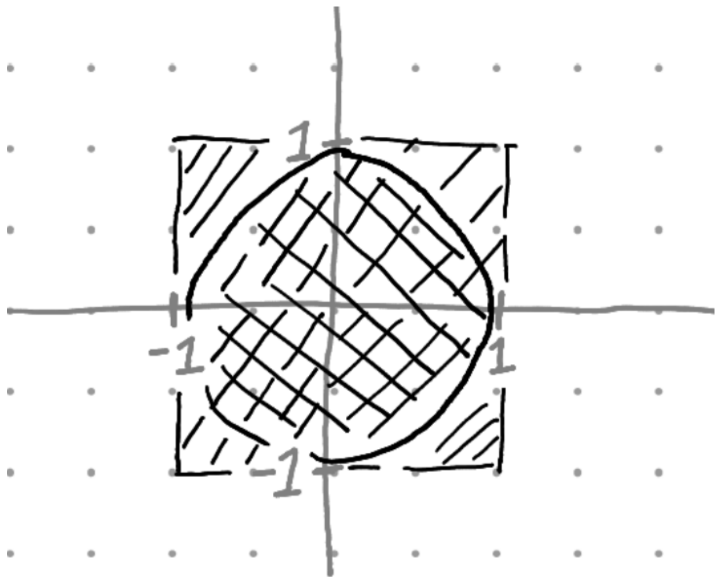
```
def uniformPrime(n)
    do
        # keep sampling uniform numbers in {1, .., 20n}
        rand = uniformRandom(20n)

    # as long as the sampled is not prime
    while not isPrime(rand)

    # only if we found a prime, we return it
    return rand
```

Correctness. We used the technique of *Rejection Sampling* which guarantees the prime number output is uniform. The proof idea is very similar to our previous probability proofs.

Analysis. From *Fact A.2.12* (page 226), There are at least $n/\ln n$ primes in P_n. So the probability of a successful trial is at least $n/\ln n$. Then the number of trials is at most $\ln n/n$ in expectation by *Fact A.3.20* (page 233). Observe $\ln n/n \leq \log n/n$. Since we are given `isPrime` is $\mathcal{O}(\log^7 n)$, The whole algorithm `uniformPrime` takes at most $\mathcal{O}(\log^8 /n) \subset \mathcal{O}(\log^{10} n)$.

Ex. 7

Part I



The probability of sampling from R is $\frac{\textit{disk-area}}{\textit{square-area}} = \frac{\pi(1)^2}{2*2} \approx \frac{22}{7} \cdot \frac{1}{4} = \frac{22}{28}$.

It follows the number of iterations is at most $\frac{28}{22} = 1 + \frac{3}{11} \leq 2$.

Part II

Partially Solved.

Seemingly we just need to compute volumes of both S and R but in $20th$ dimension, and follow exactly the same recipe of *Part I*.