Homework 3

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Contents

Exercises

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(i)

Reflexive. Consider the identity bijection $\varphi : v \mapsto v$, Which trivially satisfies $\{uv\} \in E$ iff $\varphi(v)\varphi(u) \in E$.

Symmetric. Given a bijection φ, Its inverse φ^{-1} exists and clearly is a bijection too. Now we prove $\{uv\} \in E$ iff $\varphi^{-1}(u)\varphi^{-1}(v) \in E$.

Denote $\varphi^{-1}(u) = u'$ and $\varphi^{-1}(v) = v'$.

$(\rightarrow)$. We are given $\{uv\} \in E$, i.e $\{\varphi(u')\varphi(v')\} \in E$. By symmetry hypothesis it follows $\{u'v'\} \in E$, i.e $\{\varphi^{-1}(u)\varphi^{-1}(v)\} \in E$.

$(\leftarrow)$. We are given $\{\varphi^{-1}(u)\varphi^{-1}(v) \in E\}$, i.e $\{u'v'\} \in E$. By symmetry hypothesis,

$$\{\varphi(u')\varphi(v')\} \in E \tag{1}$$
$$\{\varphi \circ \varphi^{-1}(u)\varphi \circ \varphi^{-1}(v)\} \in E \tag{2}$$
$$\{uv\} \in E \tag{3}$$

Transitive. Given bijections φ_0 and φ_1, Construct a bijection $\varphi = \varphi_1 \circ \varphi_0$. Given $\{uv\} \in E$, It follows $\{\varphi_0(u)\varphi_0(v)\} \in E$, and in turn $\{\varphi_1 \circ \varphi_0(u) \circ \varphi_1 \circ \varphi_0(v)\} \in E$. The other way follows analogously.

(ii)

True.

Let X and Y be the two sets such that for any $\{uv\} \in E_G$, $u \in X$ and $v \in Y$. Consider an arbitrary $e \in H$. By *(i)*, $\varphi^{-1}(e) = \{xy\}$ where $x \in X$ and $y \in Y$. It follows $\varphi(x) \in \varphi(X)$ and $\varphi(y) \in \varphi(Y)$. So H is bipartite.

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$(\leftarrow)$. Given G is complete. Let S be any subset of vertices. Consider $G - S$ and let $u, v \in G - S$ be arbitrary. Since G is complete, $G - S$ has the edge $\{uv\}$, and hence connected. It follows S is not a cut-set.

$(\rightarrow)$. If G has no cut-set, Then for any $S \subset V$, $G - S$ is connected. Consider $S_{uv} = V - \{uv\}$. Since $G - S_{uv}$ is connected, it has some path $(u, u_1, \ldots, u_{k-1}, v)$, but the graph has only vertices u and v. Therefore $G - S_{uv}$ has the edge $\{uv\}$, and in turn G has $\{uv\}$ also. That concludes G is complete.

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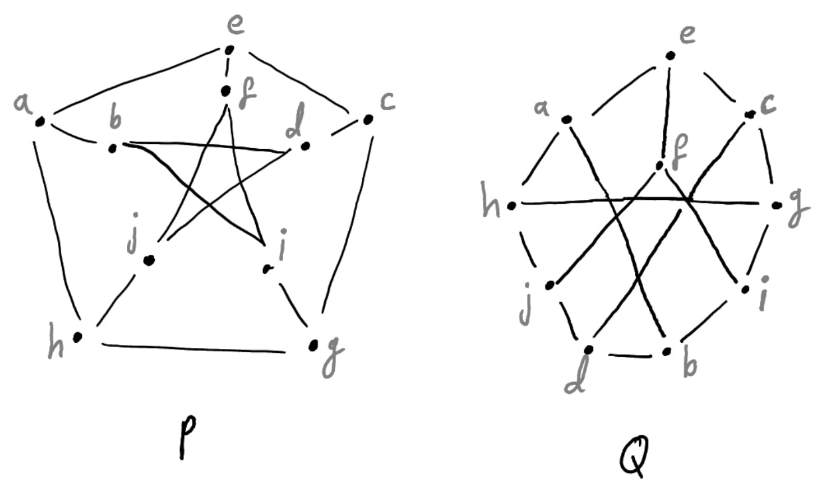
(i)

$P \cong Q$.

(ii)



(iii)

Not Solved.

Brute-forcing all possible mappings seems non-elegant.

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False. Here is a counter-example.