Homework 4

Mostafa Touny

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Contents

Exercises

1

The least path we can construct between x and y is (x) in the case of $x = y$. So $d(x, y) \geq 0$.

if $x = y$, then (x) is a valid path, concluding $d(x, y) = 0$. If $d(x, y) = 0$ then the graph contains a path $(x) = (y)$, concluding $x = y$.

Since the graph is undirected, It is clear $(v, v_1, \dots, v_k)$ is the least vv_k path if and only if $(v_k, v_{k-1}, \dots, v)$ is the least $v_k v$ path. It follows $d(x, y) = d(y, x)$.

The paths $(x, \dots, y)$ and $(y, \dots, z)$ can be concatenated to construct a new path $(x, \dots, z)$ whose length is $d(x, y) + d(y, z)$. If we considered the minimal-length xz path, Then by definition we get $d(x, z) \leq d(x, y) + d(y, z)$. $\blacksquare$.

2

(i)

By definition $ecc(v) = diam(G)$. So $d(u, v) = diam(G)$. It follows $ecc(u) = diam(G)$ as there is no longer path than the diameter.

(ii)

Fix u and v. Let u_m and v_m be the vertices corresponding to maximum paths for u and v respectively.

Assume for contradiction the difference between $d(u, u_m)$ and $d(v, v_m)$ is greater than 1. WLOG assume $d(u, u_m)$ is the greater. Then there is a path $(u, \dots u_m)$ whose length is greater than $d(v, v_m)$.

(iii)

3

(i)

Fact. $A^k(i, j)$ is the number of $v_i v_j$ paths of length k.

By definition $d(v_i, v_j)$ is the least length of any path. In other words, least k where $A^k(i, j) > 0$.

(ii)

Consider $\min\{k \mid S^k \text{ has some rows with no zeros}\}$. Those rows correspond to vertices $C(G)$.

Consider $\min\{k \mid S^k \text{ has no zeros}\}$. Rows with zeros in S^{k-1} correspond to vertices $P(G)$.

4

(i)

Fact. A connected tree of order n has exactly $n - 1$ edges.

Let n_i denote the order of ith connected tree. Then by the *Fact* ith tree has $n_i - 1$ edges. It follows the total number of edges of the forest is $\sum_{i=1}^{k}(n_i - 1) = n - k$.

(ii)

If some edge is not a bridge then a cycle shall exist, contradicting the fact a tree is acyclic.