Homework 06

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Contents

Exercises

1

($\rightarrow$). Assume connected G has an Eulerian trail. If G has also an Eulerian circuit then we are done by *theorem 3.5*. Then we have trail $t = (v_1, v_2, \dots, v_k)$ where t is not a circuit, i.e $v_1 \neq v_k$.

Since the graph is connected, We know all vertices do appear in t. For any interior vertex u in t we know it counts two edges by the definition of a trail and the prohibition of self-loops. It follows any vertex $u \neq v_1, v_k$ is of even degree, As its degree is 2 times the number of times it appears in t. Moreover v_1 counts an additional single edge since the trail starts with it, So its degree is odd. Similarly v_k is of an odd degree.

($\leftarrow$).

Case 1. All vertices have even degrees.

Then by *theorem 4.5*, G is Eulerian, So it has an Eulerian trail.

Case 2. Exactly 2 vertices have odd degrees. Call them v_1 and v_2.

Case 2.1. Edge $\{v_1, v_2\} \in E(G)$.

Define $H = G - \{v_1, v_2\}$, Removing the edge in G. Then all vertices have even degrees. By *Theorem 3.5* H contains an Eulerian circuit $c = (v_1, u_1, u_2, \dots, v_1)$. It follows G has the trail $t = c + (v_1, v_2)$.

Case 2.2. Edge $\{v_1, v_2\} \notin E(G)$.

Define $H = G + \{v_1, v_2\}$, Adding the edge to G. Then all vertices have even degrees. By *Theorem 3.5* the graph contains an Eulerian circuit $c = (u_1, u_2, \dots, u_{k-1}, u_1)$, Since it covers all the edges, WLOG (v_1, v_2) is contained in it.

Construct a new trail t from the circuit c, starting at v_2 and ending at v_1. Clearly t is a trail in G. ■

1

Proof Idea. (Lazy to typeset a complete proof).

Odd numbers do start from 1. Then every edge is contained in some cycle. Pick some edge and its corresponding cycle. Select another edge and another cycle. By connectivity there is a path p between the two cycles. Select some edge e in p and through its third cycle, Combine the three cycles to form a larger cycle. Repeat the process until the whole graph is covered.