Homework 08

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Contents

Exercises

Lemma 1.

- If a graph is a subdivision of $K_{3,3}$, then it contains 3 vertices of *deg* 3.
- If a graph is a subdivision of K_5, then it contains 5 vertices of *deg* 4.

Corollary 2. If a graph neither has a subgraph with 3 vertices of *deg* 3, nor a subgraph with 5 vertices of *deg* 4, Then it is planar.

1

(1). $K_{r,s}$ is not planar $\leftrightarrow r \geq 3$ and $s \geq 3$.

($\leftarrow$). Clear as $K_{r,s}$ shall contain $K_{3,3}$ as a subgraph.

($\rightarrow$). We show the contrapositive. Given $r < 3$ or $s < 3$, The premise of *Corollary 2* is clearly satisfied, and hence $K_{r,s}$ is planar.

(2). $K_{r_1,r_2,\ldots,r_k}$ is not planar $\leftrightarrow$ there are consecutive $r_i \geq 3$ and $r_{i+1} \geq 3$.

Similar to *(1)*.

2

(a).

($\leftarrow$) Trivial as by definition $\Big\{\{x,y\},\{y,z\}\Big\}$ is a subdivision of $\Big\{\{x,z\}\Big\}$, resulting in graph G from H.

Fact 3. For any new vertices a and b resulting from subdivisions, incident to existing vertices x and z respectively, $a \neq b$, prohibiting $\Big\{\{x,a\},\{a,z\}\Big\}$.

Fact 4. Newly added edges from subdivisions are incident to a newly added vertex and to an existing vertex.

($\rightarrow$). We are given G is a subdivision of H. Call the sequence of subdivisions $S_1, S_2, \ldots, S_k$, and the corresponding graphs $G_0, G_1, \ldots, G_k$, whereby $G_0 = H$ and $G_k = G$.

Necessarily some S_i is on edge $\{xz\}$. Otherwise by *Fact 3* we won't have $\Big\{\{x,y\},\{y,z\}\Big\}$ in G. Clearly a subdivision on $\{xz\}$ is unique, Call it S_m.

By *Fact 4* a newly added edge cannot be incident to two existing vertices. It follows edge $\{xz\}$ is not in any G_j for $j \geq m$. In particular $G_k = G$ has no edge $\{xz\}$. ■

(b).

Given a graph G_i, If $\{xy\}, \{yz\} \in E(G_i)$ and $\{xz\} \notin E(G_i)$, Construct $G_{i+1} = G_i - \{xy\} - \{yz\} + \{xz\}$. Now given an arbitrary graph G keep applying this procedure until it can no longer be applied. Since the graph has finite edges, The process terminates. So we get $G_0, G_1, \dots, G_k$ where $G_0 = G$ and G_{i+1} is a subdivision of G_i. Set $H = G_k$. Now H is not a proper subdivision of any other graph by *(a)* ■

(c).

Given a graph H is not a subdivision of any other graph, and following the same line of reasoning of *(b)*, For any $\{xy\}, \{yz\} \in E(H)$, $\{xz\} \in E(H)$. Assuming H is connected,

Case $\Delta(G) = 0$. Then $G = K_1$.

Case $\Delta(G) = 1$. Then $G = K_2$.

Case $\Delta(G) = 2$. Then $G = K_3$, As we must have a vertex of *deg* 2, and by hypothesis the endpoints are connected by an edge. The existince of any additional edge shall construct a vertex with *deg* 3.

If H is not connected, Then each of its components are either K_1, K_2, or K_3. ■

3

(a).

$$\begin{aligned} 2m &= \sum deg\, v \\ &= 24 \cdot 3 \\ m &= 12 \cdot 3 = 36 \end{aligned}$$

$$\begin{aligned} n - m + r &= 2 \\ 12 - 36 + r &= 2 \\ r &= 26 \end{aligned}$$

(b). Assume for contradiction $\delta(G) \geq 5$. Then

$$m = \frac{1}{2} \sum deg\, v \geq \frac{5}{2} n$$

By *theorem 5.15*

$$m \leq 3(n-2) = 3n - 6$$

It suffices to show $3n - 6 < \frac{5}{2}n$ to obtain a contradiction. Given $n < 12$,

$$\begin{aligned} n + 5n - 12 &< 12 + 5n - 12 \\ 6n - 12 &< 5n \\ 3n - 6 &< \frac{5}{2}n \quad \blacksquare \end{aligned}$$

4

Not-planar as it contains a subdivision of K_5.