Homework 09

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Contents

Exercises

1

Any C_n with vertices $\{v_1, v_2, \dots, v_n\}$ is *3-colorable* by $k = \{1, 2, 3, 1, 2, 3, 1, \dots\}$.

Any C_n, where n is even, is *2-colorable* by $k = \{1, 2, 1, 2, 1, \dots\}$. Observe $k(v_i) = 1$ if i is odd and $k(v_i) = 2$ if i is even. Particularly $k(v_1) \neq k(v_n)$.

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Fact 1. Colouring a graph is equivalent to partitioning it into independent subsets.

Fact 2. Since a colouring exists of any graph, It follows there exist independent subsets P_i of $V(G)$ such that $\bigcup P_i = V(G)$, and $P_i \cap P_j = \phi$ for $i \neq j$.

Fact 3. $|P_i| \leq \alpha(G)$.

(a).

Given a graph, Colour it with $\chi(G)$ colours. By *Fact 1* there exists equivalent independent subsets $P_1, P_2, \dots, P_{\chi(G)}$. By *Fact 3*

$$|G| = \sum_{i=1}^{\chi(G)} |P_i| \leq \sum_{i=1}^{\chi(G)} \alpha(G) = \chi(G) \cdot \alpha(G) \quad \blacksquare$$

(b).

Given a graph G of order n, label its vertices by $v_1, v_2, \dots, v_{\alpha(G)}, v_{\alpha(G)+1}, \dots, v_n$.

We know the greedy algorithm constructs a valid colouring for G. So the number of colours will be at least χG.

By independency, $v_1, v_2, \dots, v_{\alpha(G)}$ counts one colour. It follows $v_{\alpha(G)+1}, \dots, v_n$ contains at least $\chi_G - 1$ vertices to fulfill remaining colours. Therefore

$$|G| = n \geq \alpha(G) + \chi(G) - 1$$

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Fact 4. For a possibly disconnected graph G, $\Delta(G) = \max\{\Delta(G_i) \mid G_i \text{ is a component in } G\}$, and $\chi(G) = \max\{\chi(G_i) \mid G_i \text{ is a component in } G\}$.

The condition is that the graph G contains a component $G_i = C_n$ or $G_i = K_n$ whereby $\Delta(G_i) = \Delta(G)$.

Sufficient.

Case 1. If the graph contains a component K_n where $\Delta(K_n) = \Delta(G)$, Then $\chi(G) = \chi(K_n)$. To see why, Assume for the sake of contradiction $\chi(G_i) > \chi(K_n)$ for some component G_i, Then

$$\chi(G_i) > \chi(K_n) = \Delta(K_n) + 1 = \Delta(G) + 1 \geq \Delta(G_i) + 1$$

Case 2. If the graph contains a component C_n for odd n where $2 = \Delta(C_n) = \Delta(G)$ then also $3 = \chi(C_n) = \chi(G)$. To see why, Symmetrically to *Case 1*, Assuming $\chi(G_i) > \chi(C_n)$ for any component G_i yields $\chi(G_i) > \Delta(G_i) + 1$.

Now for both *Case 1* and *Case 2*, By *Brooke* we know the bound is sharp for component K_n and C_n. That implies it is sharp for the whole graph also. For example $\chi(K_n) = \Delta(K_n) + 1$ implies $\chi(G) = \Delta(G) + 1$.

Necessary.

We show the contrapositive. Assume for any component G_i, If $\Delta(G_i) = \Delta(G)$ then neither $G_i = K_n$ nor $G_i = C_n$. By *Brooke*

$$\chi(G_i) < \Delta(G_i) + 1 = \Delta(G) + 1 \quad (1)$$

For components G_j where $\Delta(G_j) < \Delta(G)$, We get

$$\chi(G_j) \leq \Delta(G_j) + 1 < \Delta(G) + 1 \quad (2)$$

From (1) and (2), It follows $\chi(G) < \Delta(G) + 1$, i.e the bound is not sharp. ■

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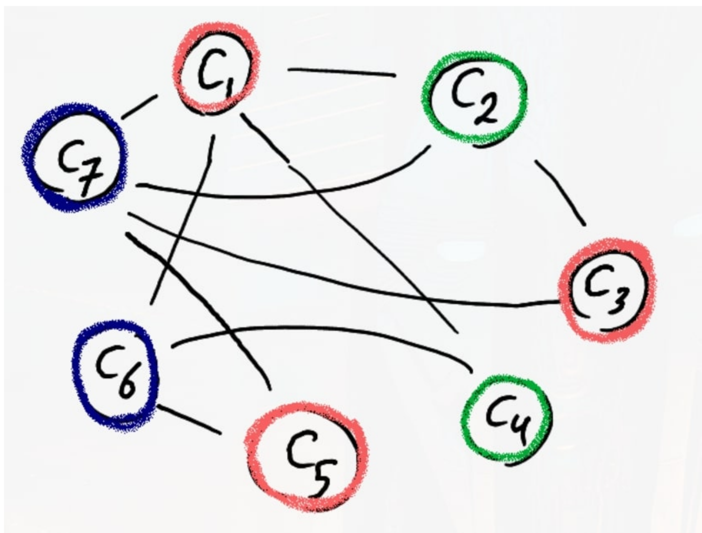
We reduce the problem to graph colouring as follows:

- Vertices of the graph are the committees.
- Two distinct committees C_i and C_j are connected if and only if some member is in both of them.
- Number of meetings is the number of colours.

The resulting graph is



So 3 meetings are needed.