Graph Theory
Homework Series 10
In Groups of 2-4 Students

Keywords: Chromatic Polynomials, Hall's Theorem.

Exercise 1. *Let G be any graph of order n.*

(i) Prove that $P_G(k)$ is a polynomial in the variable k of degree n, i.e.

$$P_G(k) = a_0 + a_1 k + \cdots + a_{n-1}k^{n-1} + a_n k^n.$$

(ii) Prove that with $P_G(k)$ as in (i), we always have $a_0 = 0$ and $a_n = 1$.

3+3 points.

Exercise 2. *Let G be any graph and M a matching for G. Prove the following.*

(i) Every maximum matching is a maximal matching.

(ii) Every perfect matching is a maximum matching.

(iii) Every graph has a maximum matching.

(iv) If G has a perfect matching, then every maximum matching is a perfect matching and $|G|$ is even.

2+2+2+3 points.

Exercise 3 (Hall's Marriage Theorem)**.** *(1) Let G be a bipartite graph with nonempty parts X and Y. Show that X is matched into Y if and only if there exists an injective function $f : X \to Y$ such that $\{x, f(x)\} \in E(G)$ for all $x \in X$.*

(2) Prove Theorem 7.15 from the notes, i.e. show that for a family $\mathcal{F} := \{S_1, S_2, \ldots, S_k\}$ of nonempty sets, there is a system of distict representatives if and only if for any $I \subseteq \{1, 2, \ldots, k\}$ we have

$$|I| \leq |\bigcup_{i \in I} S_i|.$$

2+3 points.