Chapter 1

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Exercises

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We prove by strong induction the statement: For any $m \geq 0$, If G is a graph with m edges and n vertices, Then

- $P_G(k) = a_0 + a_1 k^1 + \cdots + a_n k^n$, For some polynomial representation.
- $a_0 = 0$
- $a_n = 1$

Base case. $m = 0$. Then $G = E_n$, the empty graph with n vertices. By combinatorics we know $P_G(k) = k^n = 0 + (1)k^n$.

Induction hypothesis. Assume the statement holds for any graph with at most m edges for $m \geq 0$.

Induction step. For a graph G with $m+1$ edges and n vertices, Note G has some edge e and the graphs $G - e$ and G/e have at most m edges. By *Birkhoff* and the *induction hypothesis*,

$$\begin{aligned} P_G(k) &= P_{G-e}(k) - P_{G/e}(k) \\ &= (0 + a_1 k^1 + \cdots + (1)k^n) - (0 + a'_1 k^1 + \cdots + (1)k^{n-1}) \\ &= 0 + (a_1 - a'_1)k^1 + \cdots + (a_{n-1} - 1)k^{n-1} + (1)k^n \quad \blacksquare \end{aligned}$$

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(i). For a maximum matching M, if adding an additional edge results in a valid matching, then M won't be maximum. Hence it is maximal.

(ii). For a perfect matching M let $k = |M|$. If there were a matching with more than k edges, then that matching shall cover more than $2k$ vertices. That implies the graph contains more than $2k$ vertices, and as a result M does not cover all graph's vertices, so not perfect.

(iii). Consider the set $AM = \{M \mid M \text{ is a matching of } G\}$. Since a matching M is a subgraph, and there are finitely many subgraphs, it follows AM is finite. Moreover it has a maximum edge size matching.

(iv). Call the perfect matching M_0 and let $k_0 = |M_0|$. By definition it covers all vertices of the graph, and in turn G has exactly $2k_0$ vertices, Concluding $|G|$ is even.

It suffices to prove, If an arbitrary matching M is not perfect, Then it is not maximum. By definition M covers less than $2k_0$ vertices. Then M has less than k_0 edges, So $|M| < |M_0|$. Since M_0 is a valid matching, It follows M is not maximum. $\blacksquare$

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(1). This is exactly the definition of matching X into Y.

(2). We reduce it to *Hall's Marriage*. Let $X = \{S_i \mid i \in I\}$ and $Y = \bigcup_{i \in I} S_i$. Now the problem of *system of distinct representatives* is equivalent to matching hall of X into Y. In other words, Matching childs injectively to gifts is equivalent to matching S_i to representatives. By *Hall's Marriage Theorem*, This is possible if and only if $|X| \leq |N(X)|$. However, $|X| = |I|$ and $N(X) = Y$. Therefore *system of distinct representatives* is possible *iff* $|I| \leq |\bigcup_{i \in I} S_i|$.