Chapter 11

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Exercises

1

We know a maximum matching M and a minimum covering C do both exist, where $|M| = |C|$.

Observe by definition all edges $E(G)$ are exactly the edges incident to cover C. Clearly each edge distinctly contributes to the degrees sum. It follows $\sum_{v \in C} deg\ v \geq |E(G)|$.

Hence

$$\Delta(G)\ |M| = \Delta(G)\ |C| = \sum_{v \in C} \Delta(G) \geq \sum_{v \in C} deg\ v \geq |E(G)|$$

2

We reduce the problem to *theorem 7.15* and show for any $I \subset \{1, 2, \ldots, r\}$, $|I| \leq |\bigcup_i S_i|$.

Take a subset $I = \{\pi_1, \pi_2, \ldots, \pi_m\} \subset \{1, 2, \ldots, r\}$ where $0 \leq m \leq r$. If $m = 0$ then trivially $|I| = 0 = |\phi| = |\bigcup_i S_i|$. Consider $m \geq 1$. Since cardinalities of S_{π_i} are all positive and distinct, By the *Pigeon Hole Principle*, m subsets cannot be assigned cardinalities between 1 and $m - 1$. Therefore there exists a subset S with $|S| \geq m$. It follows $|\bigcup_i S_{\pi_i}| \geq |S| \geq m = |I|$.

4

Claim. $R(a, b) \leq R(b, a)$

We know there is a $K_{R(b,a)}$ such that any edge 2-coloring either has a red K_b or a green K_a as subgraphs.

But if it has a red K_b, then trivially it has a green k_b, and if it has a green K_a, then trivially it has a red K_a.

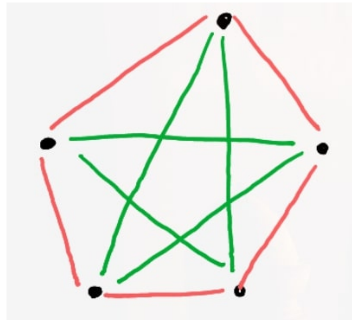
Then there is a $K_{R(b,a)}$ such that any edge 2-coloring either has a red K_a or a green K_b. Hence, $R(a, b) \leq R(b, a)$.

By symmetry $R(b, a) \leq R(a, b)$, Concluding $R(a, b) = R(b, a)$. ■

5

Claim. $R(3, 3) > 5$.

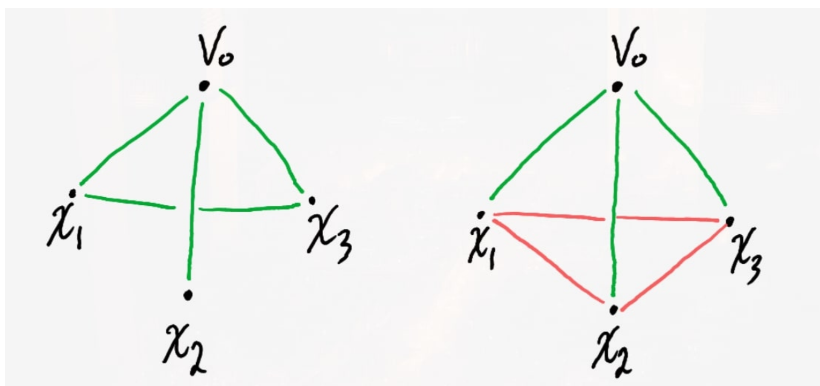
The following edge colouring of K_5 neither has a red K_3 nor a green K_3.



Observe taking any 3 vertices, 2 of them will be adjacent by a red edge, and 2 of them will be adjacent by a green edge.

Claim. $R(3,3) \leq 6$.

Fix vertex v_0, and consider any edge colouring. Since *deg* $v_0 = 5$, Necessarily 3 of which are of the same colour. WLOG assume the colour is *green*. Call those 3 vertices x_1, x_2, x_3. As in the following figure we have two cases:



Either x_1, x_2, x_3 form a red K_3, or some edge of them is green, for example edge $\{x_1, x_3\}$. Therefore, Either the graph contains a *red* K_3 or a *green* K_3. ■