# Problem-Set 01

Mostafa Touny

June 18, 2022

## Contents

# Problem. 1

**Definition. 1** *simplest-fraction*
We call $\frac{x}{y}$ a *simplest-fraction* when it is in its simplest form. i.e cannot be reduced by eliminating a common fraction.

**Fact. 2** If there are no common factors between $x$ and $y$, Then $\frac{x}{y}$ is a *simplest-fraction*.

**Lemma. 3** If $\frac{x}{y}$ is a *simplest-fraction*, Then so is $\frac{x^2}{y^2}$
Observe any common factor among the numerator and denominator is going to necessarily divide both $x$ and $y$.

**Theorem. 4** Main Problem
It's possible to set $\sqrt{\frac{m}{n}} = \frac{a}{b}$ where $\frac{a}{b}$ is a *simplest-fraction*. On $\frac{m}{n} = \frac{a^2}{b^2}$ It follows by *fact 2* and *lemma 3*, Both $\frac{m}{n}$ and $\frac{a^2}{b^2}$ are *simplest-fractions*. By uniqueness of such forms, $m = a^2$ and $n = b^2$. QED

# Problem. 2

Assume for the sake of contradiction, There's an order $<$ defined in the complex field, Which turns it into an ordered field.

By definition, It's an ordered set also, and hence the following fact applies to it: For any $s, r \in F$ exactly one of $(i)\, s = r$, $(ii)\, s < r$, $(iii)\, s > r$ is true. Particularly we have exactly one of the following cases to be true:

**(i)** $\sqrt{-1} = 0$
Then $-1 = \sqrt{-1}\sqrt{-1} = 0 \cdot 0 = 0$. A contradiction.

**(ii)** $\sqrt{-1} > 0$
Then $-1 = \sqrt{-1}\sqrt{-1} > \sqrt{-1} \cdot 0 = 0$. Also $1 < 0$ and $\sqrt{-1} = \sqrt{-1} \cdot 1 < \sqrt{-1} \cdot 0 = 0$. A contradiction.

**(iii)** $\sqrt{-1} < 0$
Then $-1 = \sqrt{-1}\sqrt{-1} > \sqrt{-1} \cdot 0 = 0$. Also $1 < 0$ and $\sqrt{-1} = \sqrt{-1} \cdot 1 > \sqrt{-1} \cdot 0 = 0$. A contradiction.

# Problem. 3

The proof of complex numbers being an ordered set follows immediately by the obvious/natural properties of real numbers' order and enumerating cases.

A counter-example is given to the claim, that the orderd-set of complex numbers have the least-upper-bound property. Let $E = \{\ (1/x,\ y) \mid x > 1\ \}$, which is clearly bounded by any element of the set $B = \{\ (1, y) \mid y \in \mathcal{R}\ \}$. But set $B$ has no least element.

# Problem. 4

## a

$f(0) = f(0+0) = f(0) + f(0)$. Then $f(0) - f(0) = f(0) + f(0) - f(0)$, implying $0 = f(0)$.

$f(1) = f(1 \cdot 1) = f(1) \cdot f(1)$. Let $x = f(1)$, Then $x = x^2$ which implies $x(x-1) = 0$, and finally either $x = 0$ or $x = 1$.

## b

**Lemma. 1** $f(n) = nf(1)$
$f(n) = f(n-1+1) = f(n-1) + f(1)$, implying $f(n) = f(0) + n \cdot f(1)$.

**Lemma. 2** $f(n/m) = (n/m)f(1)$
$f(n/m) = f(n \cdot 1/m) = n \cdot f(1/m) \cdot f(1)$. But $f(1) = f(1/m + 1/m \ldots + 1/m) = f(1/m) + f(1/m) + \ldots + f(1/m) = m \cdots f(1/m)$, which leads to $f(1/m) = f(1)/m$.

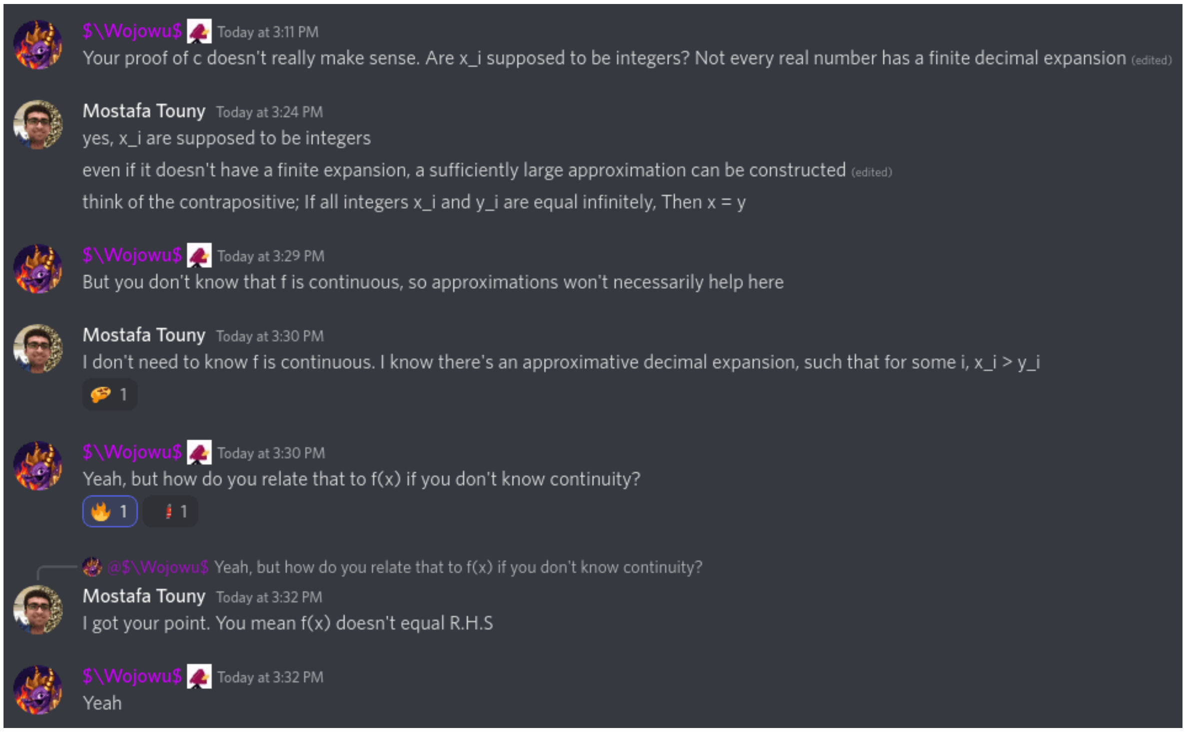
The final conclusion follows immediately by cases of $f(1)$ being equal to 0 or 1.

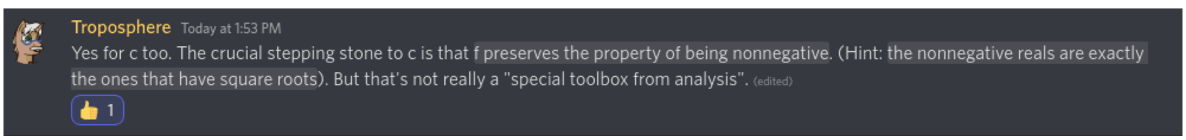
## c

**Lemma. 1** $f(x) \geq 0$ if $x \geq 0$
Since $x$ is a non-negative, we know $\sqrt{n}$ exists. Observe $f(x) = f(\sqrt{x} \cdot \sqrt{x}) = f(\sqrt{x}) \cdot f(\sqrt{x})$. But any square cannot be a negative number.

**Theorem. 2** Main Problem
If $x > y$, Then $x - y > 0$. By *Lemma. 1*, $f(x-y) \geq 0$. But $f(x-y) = f(x) + f((-1) \cdot y) = f(x) - f(y)$.

**Note.** This problem was solved by the aid of good friends. See the coversation below:




**c**

If $x > y$, Then the decimal expansion of

$$x = \frac{x_0}{10^0} + \frac{x_1}{10^1} + \frac{x_2}{10^2} + \ldots + \frac{x_k}{10^k}$$
$$y = \frac{y_0}{10^0} + \frac{y_1}{10^1} + \frac{y_2}{10^2} + \ldots + \frac{y_k}{10^k}$$

has some $x_i > y_i$. Assume $i$ is the least such index.

It follows by $b$

$$f(x) = f(1)\left[f(\frac{x_0}{10^0}) + f(\frac{x_1}{10^1}) + f(\frac{x_2}{10^2}) + \ldots + f(\frac{x_k}{10^k})\right] = f(1)\left[\frac{x_0}{10^0} + \frac{x_1}{10^1} + \frac{x_2}{10^2} + \ldots + \frac{x_k}{10^k}\right]$$
$$f(y) = f(1)\left[f(\frac{y_0}{10^0}) + f(\frac{y_1}{10^1}) + f(\frac{y_2}{10^2}) + \ldots + f(\frac{y_k}{10^k})\right] = f(1)\left[\frac{y_0}{10^0} + \frac{y_1}{10^1} + \frac{y_2}{10^2} + \ldots + \frac{y_k}{10^k}\right]$$

Considering both cases of $f(1)$ from $a$, The inequality $f(x) \geq f(y)$ follows.

Note even if $x$ or $y$ were periodic, i.e the expansion does not end, some sufficiently large $k$ would still exist fulfilling our construction.

## d

**Case. 1** $f(1) = 0$
For any real number $x$ we can pick-up two rational numbers $q_0$ and $q_1$ such that $q_0 \leq x \leq q_1$, But we know $f(q_0) = f(q_1) = 0$ from $b$, and $f(q_0) \leq f(x) \leq f(q_1)$ from $c$.

**Case. 2** $f(1) = 1$
For any real-number $x$, We know there are rational numbers $q_{1a}, q_{2a}, \ldots$ which arbitrarily get closer to $x$ from above, and similarly we know there are rational numbers $q_{1b}, q_{2b}, \ldots$ which arbitrarily get closer to $x$ from below. So we have $q_{ib} \leq x \leq q_{ia}$ for $i = 1, 2, \ldots$.

From $c$ we get $f(q_{ib}) \leq f(x) \leq f(q_{ia})$, and by $b$ $q_{ib} \leq f(x) \leq q_{ia}$, which suffices to prove $f(x) = x$.