# Problem-Set 01 

Mostafa Touny

June 18, 2022

## Contents

Problem. 1 2
Problem. 2 2
Problem. 3 3
Problem. 4 5

## Problem. 1

Definition. 1 simplest-fraction
We call $\frac{x}{y}$ a simplest-fraction when it is in its simplest form. i.e cannot be reduced by eliminating a common fraction.

Fact. 2 If there are no common factors between $x$ and $y$, Then $\frac{x}{y}$ is a simplest-fraction.
Lemma. 3 If $\frac{x}{y}$ is a simplest-fraction, Then so is $\frac{x^{2}}{y^{2}}$
Observe any common factor among the numerator and denominator is going to necessarily divide both $x$ and $y$.

Theorem. 4 Main Problem
It's possible to set $\sqrt{\frac{m}{n}}=\frac{a}{b}$ where $\frac{a}{b}$ is a simplest-fraction. On $\frac{m}{n}=\frac{a^{2}}{b^{2}}$ It follows by fact 2 and lemma 3, Both $\frac{m}{n}$ and $\frac{a^{2}}{b^{2}}$ are simplest-fractions. By uniqueness of such forms, $m=a^{2}$ and $n=b^{2}$. QED

## Problem. 2

Assume for the sake of contradiction, There's an order < defined in the complex field, Which turns it into an ordered field.

By definition, It's an ordered set also, and hence the following fact applies to it: For any $s, r \in F$ exactly one of $(i) s=r$, (ii) $s<r$, (iii) $s>r$ is true. Particularly we have exactly one of the following cases to be true:
(i) $\sqrt{-1}=0$

Then $-1=\sqrt{-1} \sqrt{-1}=0 \cdot 0=0$. A contradiction.
(ii) $\sqrt{-1}>0$

Then $-1=\sqrt{-1} \sqrt{-1}>\sqrt{-1} \cdot 0=0$. Also $1<0$ and $\sqrt{-1}=\sqrt{-1} \cdot 1<\sqrt{-1} \cdot 0=0$. A contradiction.
(iii) $\sqrt{-1}<0$

Then $-1=\sqrt{-1} \sqrt{-1}>\sqrt{-1} \cdot 0=0$. Also $1<0$ and $\sqrt{-1}=\sqrt{-1} \cdot 1>\sqrt{-1} \cdot 0=0$. A contradiction.

## Problem. 3

The proof of complex numbers being an ordered set follows immediately by the obvious/natural properties of real numbers' order and enumerating cases.

A counter-example is given to the claim, that the orderd-set of complex numbers have the least-upper-bound property. Let $E=\{(1 / x, y) \mid x>1\}$, which is clearly bounded by any element of the set $B=\{(1, y) \mid y \in \mathcal{R}\}$. But set $B$ has no least element.

## Problem. 4

a
$f(0)=f(0+0)=f(0)+f(0)$. Then $f(0)-f(0)=f(0)+f(0)-f(0)$, implying $0=f(0)$.
$f(1)=f(1 \cdot 1)=f(1) \cdot f(1)$. Let $x=f(1)$, Then $x=x^{2}$ which implies $x(x-1)=0$, and finally either $x=0$ or $x=1$.

## b

Lemma. $1 \quad f(n)=n f(1)$
$f(n)=f(n-1+1)=f(n-1)+f(1)$, implying $f(n)=f(0)+n \cdot f(1)$.
Lemma. $2 f(n / m)=(n / m) f(1)$
$f(n / m)=f(n \cdot 1 / m)=n \cdot f(1 / m) \cdot f(1)$. But $f(1)=f(1 / m+1 / m \ldots+1 / m)=$ $f(1 / m)+f(1 / m)+\ldots+f(1 / m)=m \cdots f(1 / m)$, which leads to $f(1 / m)=f(1) / m$.

The final conclusion follows immediately by cases of $f(1)$ being equal to 0 or 1 .

## C

Lemma. $1 \quad f(x) \geq 0$ if $x \geq 0$
Since $x$ is a non-negative, we know $\sqrt{n}$ exists. Observe $f(x)=f(\sqrt{x} \cdot \sqrt{x})=f(\sqrt{x})$. $f(\sqrt{x})$. But any square cannot be a negative number.

Theorem. 2 Main Problem
If $x>y$, Then $x-y>0$. By Lemma. $1, f(x-y) \geq 0$. But $f(x-y)=f(x)+f((-1) \cdot y)=$ $f(x)-f(y)$.

Note. This problem was solved by the aid of good friends. See the coversation below:

## ATody at 3. 1 PM

Your proof of c doesn't really make sense. Are x_i supposed to be integers? Not every real number has a finite decimal expansion (edited)
Mostafa Touny Today at 3:24 PM
yes, x_i are supposed to be integers
even if it doesn't have a finite expansion, a sufficiently large approximation can be constructed (edited)
think of the contrapositive; If all integers x _i and y _i are equal infinitely, Then $\mathrm{x}=\mathrm{y}$

## AT Tody at 329 PM

But you don't know that $f$ is continuous, so approximations won't necessarily help here

Mostafa Touny Today at 3:30 PM
I don't need to know f is continuous. I know there's an approximative decimal expansion, such that for some $\mathrm{i}, \mathrm{x}_{-} \mathrm{i}>\mathrm{y}_{-} \mathrm{i}$

1. Today at $3: 30 \mathrm{PM}$

Yeah, but how do you relate that to $\mathrm{f}(\mathrm{x})$ if you don't know continuity?
(4) 11
8. asNogns Yeah, but how do you relate that to $f(x)$ if you don't know continuity?

Mostafa Touny Today at 3:32 PM
I got your point. You mean $f(x)$ doesn't equal R.H.S

## ATody at 322 PM

Yeah

## Troposphere Today at 1:53 PM

Yes for c too. The crucial stepping stone to c is that f preserves the property of being nonnegative. (Hint: the nonnegative reals are exactly the ones that have square roots). But that's not really a "special toolbox from analysis". (editec)
$\downarrow 1$

## C

If $x>y$, Then the decimal expansion of

$$
\begin{aligned}
& x=\frac{x_{0}}{10^{0}}+\frac{x_{1}}{10^{1}}+\frac{x_{2}}{10^{2}}+\ldots+\frac{x_{k}}{10^{k}} \\
& y=\frac{y_{0}}{10^{0}}+\frac{y_{1}}{10^{1}}+\frac{y_{2}}{10^{2}}+\ldots+\frac{y_{k}}{10^{k}}
\end{aligned}
$$

has some $x_{i}>y_{i}$. Assume $i$ is the least such index.
It follows by $b$
$f(x)=f(1)\left[f\left(\frac{x_{0}}{10^{0}}\right)+f\left(\frac{x_{1}}{10^{1}}\right)+f\left(\frac{x_{2}}{10^{2}}\right)+\ldots+f\left(\frac{x_{k}}{10^{k}}\right)\right]=f(1)\left[\frac{x_{0}}{10^{0}}+\frac{x_{1}}{10^{1}}+\frac{x_{2}}{10^{2}}+\ldots+\frac{x_{k}}{10^{k}}\right]$
$f(y)=f(1)\left[f\left(\frac{y_{0}}{10^{0}}\right)+f\left(\frac{y_{1}}{10^{1}}\right)+f\left(\frac{y_{2}}{10^{2}}\right)+\ldots+f\left(\frac{y_{k}}{10^{k}}\right)\right]=f(1)\left[\frac{y_{0}}{10^{0}}+\frac{y_{1}}{10^{1}}+\frac{y_{2}}{10^{2}}+\ldots+\frac{y_{k}}{10^{k}}\right]$
Considering both cases of $f(1)$ from $a$, The inequality $f(x) \geq f(y)$ follows.
Note even if $x$ or $y$ were periodic, i.e the expansion does not end, some sufficiently large $k$ would still exist fulfilling our construction.
d

Case. $1 \quad f(1)=0$
For any real number $x$ we can pick-up two rational numbers $q_{0}$ and $q_{1}$ such that $q_{0} \leq$ $x \leq q_{1}$, But we know $f\left(q_{0}\right)=f\left(q_{1}\right)=0$ from $b$, and $f\left(q_{0}\right) \leq f(x) \leq f\left(q_{1}\right)$ from $c$.

Case. $2 \quad f(1)=1$
For any real-number $x$, We know there are rational numbers $q_{1 a}, q_{2 a}, \ldots$ which arbitrarily get closer to $x$ from above, and similarly we know there are rational numbers $q_{1 b}, q_{2 b}, \ldots$ which arbitrarily get closer to $x$ from below. So we have $q_{i b} \leq x \leq q_{i a}$ for $i=1,2, \ldots$.

From $c$ we get $f\left(q_{i b}\right) \leq f(x) \leq f\left(q_{i a}\right)$, and by $b q_{i b} \leq f(x) \leq q_{i a}$, which suffices to prove $f(x)=x$.

