# Problem-Set 03 

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## Contents

Problem. 1 2
Problem. 2 4
Problem. 3 5
Problem. 4 5

## Problem. 1

The required conditions follow naturally as:

- $d^{\prime}(x, x)=\sqrt{d(x, x)}=\sqrt{0}=0$.
- If $d(x, y)>0$ then $d^{\prime}(x, y)>0$ as the square root of non-zero is non-zero. Otherwise $0^{2}=0$ contradicting the fact $d^{\prime}(x, y)>0$.
- $d^{\prime}(x, y)=\sqrt{d(x, y)}=\sqrt{d(y, x)}=d^{\prime}(y, x)$.
- $d^{\prime}(x, y)=\sqrt{d(x, y)} \leq \sqrt{d(x, r)+d(r, y)} \leq \sqrt{d(x, r)}+\sqrt{d(r, y)}=d^{\prime}(x, r)+$ $d^{\prime}(r, y)$.

For an arbitrary open-set of $d,\{y \mid d(x, y)<r\}$ there is an equivalent open-set of $d^{\prime}$, $\left\{y \mid d^{\prime}(x, y)<\sqrt{r}\right\}$. For an arbitrary open-set of $d^{\prime},\left\{y \mid d^{\prime}(x, y)<r\right\}$, there is an equivalent open-set of $d,\left\{y \mid d(x, y)<r^{2}\right\}$.

Note. Some good friends assisted in solving this problem.
@ @Mostafa Touny I do not understand the question. What is meant by an open-set for distanced
Adrienoid Today at $8: 45 \mathrm{PM}$
Have you defined closed sets or open sets in your course? (edited)
Or maybe open balls of some radius with respect to a distance (edited)
Q @Adrienoid Or maybe open balls of some radius with respect to a distance (edited)
Mostafa Touny Today at 8:47 PM
I am aware of this example for Euclidean space

## 1. Today at 8.48 PM

Every metric induces a topology on the set, that is the notion of open and closed subsets for that space
The question asks that for those two metrics $d$ and $d$ ', the open sets as defined by either metric are the same
Mostafa Touny Today at $8: 49 \mathrm{PM}$
I thought a subset is independent of the definition of a metric
lyra Today at $8: 50$ PM
what you learned for Euclidean spaces is the topology more generally in metric spaces -- your open sets of the form $B_{-} x(r)=\{y \mid d(x, y)$ $<r\}$ which is the open ball of radius $r$ about a point $x$. now this is with respect to the usual metric, and it's asking you to prove that $\mathrm{d}^{\prime}(\mathrm{x}$, $\mathrm{y})=\operatorname{sqrt}(\mathrm{d}(\mathrm{x}, \mathrm{y}))$ defines another metric on the space, so you have to verify the axioms, and then show that a set is open in the topology induced by the former metric if and only if it's open in the topology induced by the latter
@ @Mostafa Touny I do not understand the question. What is meant by an open-set for distanced $i$
_alraC Today at 8.51 PM
A subset $X$ is open for $d$ if for every $x$ in $X$, there is a $r>0$ such that the ball $\{y: d(x, y)<r\}$ is a subset of $X$
$\bigcirc$ Mostafa Touny I thought a subset is independent of the definition of a metric

1. Today at 8:51 PM

Whether a subset is open or not depends on the metric

## Problem. 2

Lemma. 1 For any point $p$ in $R$, There exists a smallest element in the set $\{q \in$ $E \mid q>p\}$
Assume to the contrary that no smallest element exists. Then as the set is bounded
below, the infimum exists, and is a limit point. That contradicts our hypothesis of no limit points in $E$.

Corollary. $2 E \cap R^{+}=E^{+}$has a smallest element By the above lemma set $p=0$.

Corollarly. 3 Given $x_{i} \in E^{+}$there exists a smallest element among $E^{+} \cap\left\{y \mid y>x_{i}\right\}$ By the above lemma set $p=x_{i}$.

Now we have a counting scheme on $E^{+}$. What is remaining now is to prove every element in $E$ will be hit eventually. The following lemma suffices.

Lemma. 4 there are finitely many elements in $E \cap[-|a|,|a|]$
Assuming the contrary for the sake of contradiction, We get infinite elements in $E \cap$ $[-|a|,|a|]$. Those are present in both $E$ and $[-|a|,|a|]$ by definition. Since $[-|a|,|a|]$ is compact we know any infinite subset has a limit point (Theorem 2.41, p. 40 in baby-rudin). But then we get a limit point in E. Contradiction

Similarly we can prove $E \cap R^{-}=E^{-}$is countable, and hence $E$ is countable also.
Note. 1 Professor Peng Zhou hinted the solution approach


Note. 2 Through chatting with good friends a cleaner alternative proof can be made as, "Because E has no limit points it is closed. Assume E is uncountable. Then there is an integer $n$ such that intersection with $[\mathrm{n}, \mathrm{n}+1]$ is also uncountable. This intersection is closed and bounded, thus compact. So we can take a sequence inside this intersection and it will have a convergent subsequence contradicting the assumption on limit points"

Mostafa Touny Yesterday at 11:09 PM
I conjecture the following approach: Establish an enumeration process of sequence $x_{-}$i in $E$, And prove there is a discrete minimum distance from x_i to $x_{-} \mathrm{i}+1$.
2. Consider $\mathbb{R}$ with the standard metric. Let $E \subset \mathbb{R}$ be a subset which has no limit points. Show that $E$ is at most countable. ( 3 points)
Even if my approach is correct, I feel the proof is going to be complicated, and that there's a cleaner way.
Do you think the approach I articulated is a good one or tedious as I guessed?


Crazy Carla Yesterday at $11: 12$ PM
Why are you assuming that E is countable?
Poopheeler II: Wrath of Khanway Yesterday at 11:12 PM
Do you need to assume $E$ is countable to do the enumeration x_i to begin with?
^

Mostafa Touny Yesterday at 11:12 PM
No
I would consider R+ first and show that E restricted to it has a smallest point. Otherwise a limit point would be present.
I guess my technique is clear now

FShrike on MSE Today at 12:13 AM
If I'm not mistaken, a set with no limit points is necessarily discrete (in any Hausdorff space) and the only discrete subsets of R are countable

Gal( $Q(2-a) / Q)$ Today at $1: 22$ AM
I think there's a cute way using Heine-Borel (edited)
Cal(Q(2_- )/O) Today at 1:34 AM
Because $E$ has no limit points it is closed. Assume $E$ is uncountable. Then there is an integer $n$ such that intersection with $[\mathrm{n}, \mathrm{n}+1]$ is also uncountable. This intersection is closed and bounded, thus compact. So we can take a sequence inside this intersection and it will have a convergent subsequence contradicting the assumption on limit points

O arcallerga) Because $E$ has no limit points it is closed. Assume $E$ is uncountable. Then there is an integer $n$ such that intersection with $[\mathrm{n}, \mathrm{n}+1]$ is a... geogristle Today at 3.52 AM
u gotta specify distinct elements of sequence
惑 Today at 4:53 AM

Suppose $\$ n(x)=\backslash \inf \{m \backslash \operatorname{in} \backslash b N \backslash \operatorname{mid}|B(x, 1 / m) \backslash c a p E|=1\} \$$ for $\$ x \backslash$ in $E \$$. Then $\$ n(x) \backslash i n \backslash b N \$$ and $\$\{B(x, 1 / n(x))\} \_\{x \backslash i n E\} \$$ is an open cover of $\$ E \$$. Since $\$ \backslash b R \$$ is heredetarily Lindelöf, in the sense of the link I post, there is a countable subcover. However, since this cover consists of disjoint subsets of $\$ E \$$ that contain exactly one member of $\$ E \$$, this countable subcover must be exactly the original cover and since $\$ E \$$ is in bijection with this cover, $\$ E \$$ must be countable.
\%olodox $\checkmark$ Bot Today at 4:53 AM

## Available

Suppose $n(x)=\inf \{m \in \mathbb{N}| | B(x, 1 / m) \cap E \mid=1\}$ for $x \in E$. Then $n(x) \in \mathbb{N}$
and $\{B(x, 1 / n(x))\}_{x \in E}$ is an open cover of $E$. Since R is heredetarily Lindelöf,
in the sense of the link I post, there is a countable subcover. However, since this cover consists of disjoint subsets of $E$ that contain exactly one member
of $E$, this countable subcover must be exactly the original cover and since $E$
is in bijection with this cover, $E$ must be countable.

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8% Today at 4:54 AM
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The link

## ©

Because $E$ has no limit points it is closed. Assume $E$ is uncountable. Then there is an integer $n$ such that intersection with $[n, n+1]$ is a...

## Mostafa Touny Today at 8.53 AM

E is uncountable. Then there is an integer $n$ such that intersection with $[n, n+1]$ is also uncountable
Would you recommend me a resource for this?
(5:4) Poopheeler II: Wrath of Khanway Today at 8:55 AM
ك.. Assume the negation. Then \$E\$ is the union of disjoint countable sets \$E\cap [ $\mathrm{n}, \mathrm{n}+1] \$$, and a countable union of countable sets is countable. But $\$ \mathbb{E}$ is uncountable (edited)
$\downarrow 1$

## Problem. 3

Assume for the sake of contradiction that the process does not stop after a finite number of steps. Then the sequence $x_{i}$ is infinite. Consider the infinite subset $\left\{x_{i}\right\}=S_{\delta}$; By hypothesis it has a limit point in $X$, Call it $p$. So for neighbourhood $N_{\delta / 4}(p)$, some point $q_{1} \neq p$ is in that neighbourhood. Let $r_{1}=d\left(p, q_{1}\right)$. Consider neighbourhood $N_{r_{1} / 2}(p)$; Clearly $q_{1}$ is not in it. So there is a point $q_{2} \neq q_{1}$ in it. We have now distinct points $q_{1}, q_{2} \in S$ such that $d\left(p, q_{1}\right) \leq \delta / 4$ and $d\left(p, q_{2}\right) \leq \delta / 4$. It follows $d\left(q_{1}, q_{2}\right) \leq d\left(q_{1}, p\right)+d\left(p, q_{2}\right) \leq \delta / 4+\delta / 4=\delta / 2$. But the construction of sequence $x_{i}$ stipulates every pair of points is of distance at least $\delta$. Contradiction.

It follows by the above lemma, that for any point $p$ in $X$, the distance between it and some $x_{i}$ of $S$ is strictly less than $\delta$. Therefore $p$ is covered by $N_{\delta} x_{i}$.

Now we prove $X$ is separable. We know for each $\delta=1 / n$, The corresponding subset $S_{1 / n}$ is finite. Clearly $\cup_{n} S_{1 / n}=S$ is countably infinite. It suffices to show, For a point $p \in X-S$, it can get arbitrarily close to points of $S$. Consider arbitrary $\delta>0$ and its corresponding neighbourhood $N_{\delta}(p)$.

Take $\delta^{\prime}=\delta / 2$, and $n^{\prime}>0$ such that $1 / n^{\prime}<\delta^{\prime}$. Consider $N_{\delta^{\prime}}(p)$. There are two cases. Case 1: A point $q \in S_{1 / n^{\prime}}$ is in $N_{\delta^{\prime}}(p)$, Then it is also in $N_{\delta}(p)$.
Case 2: No point $q \in S_{1 / n^{\prime}}$ is in $N_{\delta^{\prime}}(p)$. Then for any $z \in N_{\delta^{\prime}}(p)$ some point $q \in S_{1 / n^{\prime}}$ exists such that $d(z, q)<1 / n^{\prime}$. It follows $\delta=\delta / 2+\delta / 2>\delta^{\prime}+1 / n^{\prime}>d(p, z)+d(z, q) \geq$ $d(p, q)$. In other words, $q \in N_{\delta}(p)$.

## Problem. 4

Failed to solve.
Partial idea: Establish a sequence $x, f^{1}(x), f^{2}(x), f^{3}(x), \ldots$. If I proved it is finite then I am done, as it is necessarily the case $f^{k}(x)=f^{k+1}(x)$. If it is infinite then a limit point of it exists as $X$ is a compact set.

