Ch.12, Sec.3 - Rogawski & Adams' Calculus

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Contents

Ex. 3.90

90. Prove the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos\theta$, by referring to Figure 23. *Hint:* Consider the right triangle $\triangle PQR$.

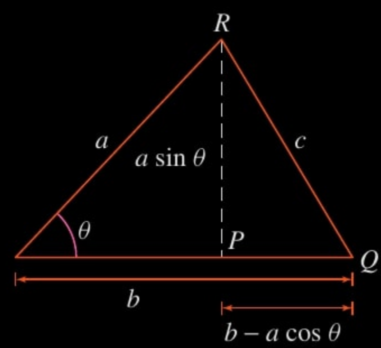

FIGURE 23

$$
\begin{aligned}
c^2 &= (a\sin\theta)^2 + (b - a\cos\theta)^2 \\
&= a^2\sin^2\theta + b^2 + a^2\cos^2\theta - 2ab\cos\theta \\
&= a^2(\sin^2\theta + \cos^2\theta) + b^2 - 2ab\cos\theta \\
&= a^2 + b^2 - 2ab\cos\theta
\end{aligned}
$$

Ex. 3.91

91. In this exercise, we prove the Cauchy–Schwarz inequality: If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors, then

$$|\mathbf{v}\cdot\mathbf{w}| \le \|\mathbf{v}\|\,\|\mathbf{w}\| \qquad \boxed{6}$$

(a) Let $f(x) = \|x\mathbf{v} + \mathbf{w}\|^2$ for x a scalar. Show that $f(x) = ax^2 + bx + c$, where $a = \|\mathbf{v}\|^2$, $b = 2\mathbf{v}\cdot\mathbf{w}$, and $c = \|\mathbf{w}\|^2$.

(b) Conclude that $b^2 - 4ac \le 0$. *Hint:* Observe that $f(x) \ge 0$ for all x.

(a). The goal is $||xv + w||^2 = ||v||^2 x^2 (2v\cdot w)x + ||w||^2$. Then

$$
\begin{aligned}
L.H.S &= (xv + w)\cdot(xv + w) \\
&= xv\cdot xv + 2xv\cdot w + w\cdot w \\
&= x^2(v\cdot v) + (2v\cdot w)x + ||w||^2 \\
&= ||v||^2x^2 + (2v\cdot w)x + ||w||^2 \\
&= R.H.S
\end{aligned}
$$

(b). We know $f(x) = ax^2 + bx + c \geq 0$. Geometrically a parabola which does not intersect the x-axis at two points. So there are no two distinct real solutions, and hence the discriminent $b^2 - 4ac \leq 0$

Ex. 3.92

92. Use (6) to prove the Triangle Inequality:

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$$

Hint: First use the Triangle Inequality for numbers to prove

$$|(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w})| \leq |(\mathbf{v} + \mathbf{w}) \cdot \mathbf{v}| + |(\mathbf{v} + \mathbf{w}) \cdot \mathbf{w}|$$

Observe $(v + w) \cdot (v + w) = (v + w) \cdot v + (v + w) \cdot w$, So hint is proven.

$$\begin{aligned} L.H.S = ||v + w||^2 &\leq ||v + w||\ ||v|| + ||v + w||\ ||w|| \\ &= ||v + w||(||v|| + ||w||) \end{aligned}$$

Thus, $||v + w|| \leq ||v|| + ||w||$

Ex. 3.93

93. This exercise gives another proof of the relation between the dot product and the angle θ between two vectors $\mathbf{v} = \langle a_1, b_1 \rangle$ and $\mathbf{w} = \langle a_2, b_2 \rangle$ in the plane. Observe that $\mathbf{v} = \|\mathbf{v}\| \langle \cos\theta_1, \sin\theta_1 \rangle$ and $\mathbf{w} = \|\mathbf{w}\| \langle \cos\theta_2, \sin\theta_2 \rangle$, with θ_1 and θ_2 as in Figure 24. Then use the addition formula for the cosine to show that

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\ \|\mathbf{w}\| \cos\theta$$

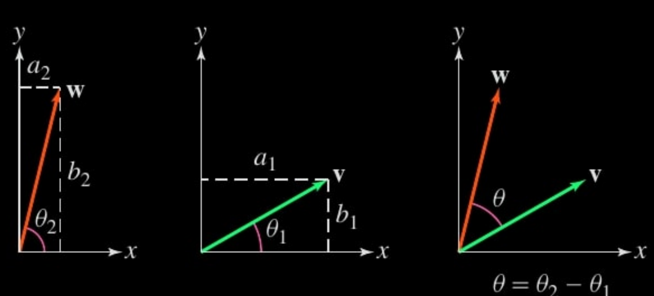

FIGURE 24

Recall the *cosine additition formula* is $\cos(a - b) = \cos a \cos b + \sin a \sin b$.

$$
\begin{aligned}
&v \cdot w \\
&\quad = ||v||(\cos\theta_1, \sin\theta_1) \cdot ||w||(\cos\theta_2, \sin\theta_2) \\
&\quad = ||v|| \cdot ||w|| \left[(\cos\theta_1, \sin\theta_1) \cdot (\cos\theta_2, \sin\theta_2)\right] \\
&\quad = ||v|| \cdot ||w|| \left[\cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1\right] \\
&\quad = ||v|| \cdot ||w|| \cos(\theta_2 - \theta_1) \\
&\quad = ||v|| \cdot ||w|| \cos(\theta) \ \text{ given } \theta = \theta_2 - \theta_1
\end{aligned}
$$

Ex. 3.94

> **94.** Let $\mathbf{v} = \langle x, y \rangle$ and
>
> $$\mathbf{v}_\theta = \langle x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta \rangle$$
>
> Prove that the angle between $\mathbf{v}$ and $\mathbf{v}_\theta$ is θ.

It suffices to show $\cos\theta = \dfrac{V \cdot V_\theta}{||V||\ ||V_\theta||}$. But $||V|| = ||V_\theta||$, Then

$$
\begin{aligned}
R.H.S &= \frac{(x^2\cos\theta + xy\sin\theta) + (-xy\sin\theta + y^2\cos\theta)}{||V||^2} \\
&= \frac{\cos\theta(x^2 + y^2)}{||V||^2} \\
&= \frac{\cos\theta\ ||V||^2}{||V||^2} \\
&= L.H.S
\end{aligned}
$$

Ex. 3.95

95. Let **v** be a nonzero vector. The angles α, β, γ between **v** and the unit vectors **i**, **j**, **k** are called the direction angles of **v** (Figure 25). The cosines of these angles are called the **direction cosines** of **v**. Prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

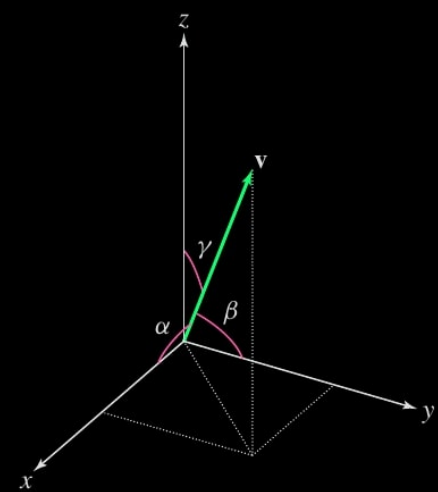

FIGURE 25 Direction angles of **v**.

Let V_x, V_y, V_z be projected vectors of V on x, y, and z axis. Then:

$$\begin{aligned}
\cos\alpha &= ||V_x||/||V|| \\
\cos\beta &= ||V_y||/||V|| \\
\cos\gamma &= ||V_z||/||V||
\end{aligned}$$

It follows

$$\begin{aligned}
&\cos^2\alpha + \cos^2\beta + \cos^2\gamma \\
&\quad = (||V_x||/||V||)^2 + (||V_y||/||V||)^2 + (||V_z||/||V||)^2 \\
&\quad = \frac{||V_x||^2 + ||V_y||^2 + ||V_z||^2}{||V||^2} \\
&\quad = \frac{||V_{x,y}||^2 + ||V_z||^2}{||V||^2} && \textit{(Pythagorean Theorem)} \\
&\quad = \frac{||V||^2}{||V||^2} && \textit{(Pythagorean Theorem)} \\
&\quad = 1
\end{aligned}$$