Chapter 02

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Problems

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(b). No. $3/2$ is not an integer.

(d). Yes. cA is a totally valid matrix for any scalar c or matrix A.

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(a). Yes.

(b). No. $\frac{1/2}{3} = \frac{1}{2}\frac{1}{3} \neq \frac{3}{2} = \frac{1}{2/3}$.

(e). No. $(2^2)^3 = 2^6 \neq 2^8 = 2^{(2^3)}$

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(c). No. $3(x^2) \neq 3^2x^2 = (3x)^2$

(d). No. Known from linear algebra.

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(a). $20 - 13 = 7$.

(b). The problem is reduced to finding x and y such that $13x = 14y + 1$. In other familiar notation from chapter 1, $13x - 14y = 1$. Clearly $13(-1) + (-14)(-1) = 1$ so $13(-1 + 14) + (-14)(-1 + 13) = 1$. Thus the inverse of 13 is 13.

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Not closed. $1 + 3 = 4$.

No inverse. $3 + x \neq 1$ for any odd integer x.

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$(ab)^3 = ababab$.

$(ab^{-2}c)^{-2} = (ab^{-2}c)^{-1}(ab^{-2}c)^{-1} = c^{-1}b^{-3}a^{-1}c^{-1}b^{-3}a^{-1}$

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Fact. x^n is an odd integer for any odd x.

Fact. The summation of two even integers is even.

We take a different perspective of the problem by the set $\{(5 \cdot 1), (5 \cdot 3), (5 \cdot 5), (5 \cdot 7)\}$ modulo $5 \cdot 8$. Upon multiplying any two elements we get the form $5 \cdot 5 \cdot x \cdot y$ where $x, y \in \{1, 3, 5, 7\}$. Think of the output of multiplication as the factor of 5 deciding the element. Observe the element is decided by $5 \cdot x \cdot y \mod 8$. For example if we knew $5 \cdot 5 \cdot 5 \cdot 1 = (5)(8 + 8 + 8 + 1)$ then we can easily deduce the output of $\mod 5 \cdot 8$ operation is $(5)(1)$.

The numbers 1, 3, 5, and 7 are all odds. So whatever x or y chosen, $5 \cdot x \cdot y$ will be odd. It follows $\mathit{odd} \mod 8 = \mathit{odd} \in \{1, 3, 5, 7\}$. To see why note $8k + \mathit{odd} = \mathit{odd}$.

Lemma. The given set is closed under the given operation.

Lemma. The identity is $5 \cdot 5 = 25$.

Observe $5 \cdot 5 \cdot x \mod 8 = 24x + x \mod 8 = x \mod 8$ since $24x \mod 8 = 0$.

Lemma. THe inverse of $5x$ is $5x$ by computation on the given elements.

Lemma. Associativity is known from integers and modulus properties.

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$(R_0)^2 = (R_{180})^2 = H^2 = V^2 = D^2 = (D')^2 = R_0$.

$(R_{90})^2 = (R_{270})^2 = R_{180}$.

So $K = \{R_0, R_{180}\}$, and $L = \{R_0, R_{180}, H, V, D, D'\}$.

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Observe the group follows the same pattern as $\mathcal{Z}_4$.

	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c

inverses. Since $ad = e$, $d = a^{-1}$. Since $bc = e$, $c = b^{-1}$.

ab = c. $ab = (cc)b = c(cb) = ce = c$.

db = a. $db = d(aa) = (da)a = ea = a$.

cd = b. $cd = c(bb) = (cb)b = eb = b$.

dc = b. $dc = (bb)c = b(bc) = be = b$.

ac = d. $d = bb = (aa)(dc) = a(ad)c = ac.$

bd = a. $bd = (dc)(bb) = d(cb)b = db = a.$

dd = c. $dd = (ac)(bb) = a(cb)b = ab = c$

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$(\leftarrow)$. Given $ab = ba$

$$
\begin{aligned}
(ab)^2 &= (ab)(ab) \\
&= a(ba)b, \text{Associativity} \\
&= a(ab)b \\
&= (aa)(bb), \text{Associativity} \\
&= a^2b^2
\end{aligned}
$$

$(\rightarrow)$. Given $(ab)^2 = a^2b^2$

$$
\begin{aligned}
(ab)^2 &= (ab)(ab) \\
&= a(ba)b \\
&= aabb \\
ba &= ab, \text{Cancellation}
\end{aligned}
$$

$(\leftarrow)$. Given $ab = ba$

$$
\begin{aligned}
(ab)^2 &= (ab)^{-1}(ab)^{-1} \\
&= b^{-1}a^{-1}b^{-1}a^{-1} \\
&= b^{-1}(ba)^{-1}a^{-1} \\
&= b^{-1}(ab)^{-1}a^{-1} \\
&= b^{-1}b^{-1}a^{-1}a^{-1} \\
&= (b)^{-2}(a)^{-2}
\end{aligned}
$$

$(\rightarrow)$. Given $(ab)^{-2} = b^{-2}a^{-2}$

$$
\begin{aligned}
(ab)^{-1}(ab)^{-1} &= b^{-1}a^{-1}b^{-1}a^{-1} \\
&= b^{-1}b^{-1}a^{-1}a^{-1} \\
a^{-1}b^{-1} &= b^{-1}a^{-1}, \text{Cancellation} \\
(ba)^{-1} &= (ab)^{-1}
\end{aligned}
$$

Now observe by the definition of inverse, if $x = y^{-1}$ then $y = x^{-1}$. Therefore $ab = [(ab)^{-1}]^{-1}$ and $ba = [(ba)^{-1}]^{-1}$, and $ab = ba$.

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Clearly $aabb = a^2b^2 = ee = e$, and $abab = (ab)^2 = e$. It follows $aabb = abab$, and by *cancellation* $ab = ba$.