$\frac{9}{10}$

Chapter 03

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Contents

Problems

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$(Q, +)$. $\{\frac{x}{2} \mid x \in Z\}$.

$(Q^*, *)$. $\{2^x \mid x \in Z^+\} \cup \{\frac{1}{2^x} \mid x \in Z^+\} \cup \{1\}$.

1 / 1

4

Consider $|x| = n$. Then $x^n = 1$ and no positive $r < n$ where $x^r = 1$. It follows

0.75 / 1

$$\begin{aligned}
(x^n)^{-1} &= (1)^{-1} \\
(x \cdot x \cdot \dots \cdot x)^{-1} &= 1 \\
x^{-1} \cdot \dots \cdot x^{-1} &= \\
(x^{-1})^n &=
\end{aligned}$$

That is $|x^{-1}| \leq n$

Analogously if $(x^{-1})^r = 1$ then $x^r = 1$. That cannot happen for $r < n$.

→ $n \leq |x^{-1}|$ Then $n = |x^{-1}|$

6(b)

Identity is $e = 0$.

$|3| = 4$. $|8| = 3$. $|11| = 12$.

1 / 1

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Fact. For any element x in any group, $x^{n+m} = x^n x^m$.

Fact. For any element x in any group, $(x^k)^m = x^{km}$.

$$\begin{aligned}
(a^4 c^{-2} b^4)^{-1} &= (b^4)^{-1}(c^{-2})^{-1}(a^4)^{-1} \\
&= (b^4)^{-1}(c^2)(a^4)^{-1} \\
&= (b^7 b^{-3})^{-1}(c^2)(a^6 a^{-2})^{-1} \\
&= (b^{-3})^{-1}(c^2)(a^{-2})^{-1} \\
&= b^3 c^2 a^2
\end{aligned}$$

b / 1

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We naively construct all possible subgroups, pruning possible branches by their properties.

Any subgroup must have the identity element. $\{R_0\}$. (+1)

$\{R_0, X\}$ is a subgroup for any reflection $X = H, V, D, D'$. (+4)

Considering a subgroup with R_0, X_0, X_1 for distinct reflections X_0, X_1 it must be the case we get rotation R_s for $s \neq 0$. So we cannot have a subgroup restricted on reflections other than the aforementioned case.

$\{R_0, R_{180}\}$. (+1)

For any subgroup with R_{90} or R_{270}, since it is closed it must contain also $\{R_0, R_{90}, R_{180}, R_{270}\}$. (+1)

For any subgroup containing $\{R_0, R_{180}, H\}$ it must contain also $\{R_0, R_{180}, H, V\}$. For any subgroup containing $\{R_0, R_{180}, V\}$ it must contain also $\{R_0, R_{180}, V, H\}$. (+1) $\frac{1}{1}$

For any subgroup containing $\{R_0, R_{180}, D\}$ it must contain also $\{R_0, R_{180}, D, D'\}$. For any subgroup containing $\{R_0, R_{180}, D'\}$ it must contain also $\{R_0, R_{180}, D, D'\}$. (+1)

For any subgroup containing R_s for $s \neq 180$ and any reflection $X = H, V, D, D'$, since it is closed, it must contain also $\{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$. (+1)

So far we counted 10 subgroups. You were asked about subgroups of order 4 specifically!

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We show the contrapositive. Assume $a^m = a^n$ for $m > n$. Then $a^m a^{-n} = a^n a^{-n}$ implying $a^{m-n} = e$, but $m - n > 0$ so a is of a finite order. $\frac{1}{1}$

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The question presumes the uniqueness of H. We won't prove it. $\frac{1.25}{2}$

$H = \{2(9k_1 + 15k_2 + 20k_3) \mid k_1, k_2, k_3 \in \mathbb{Z}\}$. $= \langle 2 \rangle$

Taking $k_1 = k_2 = k_3 = 0$ yields the identity $e = 0$. For $x \in H$ corresponding to k_i, Take $-(k_i)$ to obtain the inverse. Closed property is clear from the definition. Associativity follows from G. Odd numbers are excluded conforming to the fact H is a proper subgroup. do it!

This is not a clear description of H

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Since $e \in H$ and $e \in K$ by definition, We have $e \in H \cap K$.

if $x, y, z \in H \cap K$, then $x, y, z \in H$ and associativity follows. Associativity is inherited from G

if $x \in H \cap K$, then $-x \in H$ and $-x \in K$, and any element has an inverse. $\frac{2}{2}$

if $x, y \in H \cap K$, then $x + y \in H$ and $x + y \in K$ by properties of a group.

A trivial argument by induction shows the intersection of any number of subgroups.

consider $\{H_i : H_i \leq G\}$ $H_0 = \bigcap_{i=1}^{m} H_i$

This induction will give you only finite intersection, not arbitrary intersection!

WRITE MORE!

Transfinite induction