$\frac{6}{10}$

Chapter 07

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October 10, 2023

Contents

Problems

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Those are $\{a\langle 3\rangle \mid a \in \mathcal{Z}\} = \{\{a \pm 0, a \pm 3, a \pm 6, \dots\} \mid a \in \mathcal{Z}\}$.

$\frac{0}{0.5}$

~~The~~ Not wrong but they are only [3]

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$\frac{0.5}{0.5}$

Observe $\langle a^4\rangle = \{1, a^{4(1)}, a^{4(2)}, \dots, a^{4(14)}\}$. Then $|\langle a^4\rangle| = 15$. It follows by *theorem 7.1* (page 142), The number of distinct left cosets is $30/15 = 2$.

Hence, they are $\langle a^4\rangle$ and $a\langle a^4\rangle$

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$\frac{0.25}{0.75}$

Lazy to compute and typeset all left cosets.

H is a subgroup. Then by *theorem 7.1* (page 142), the number of left cosets of H in S_4 is $4!/4 = 3! = 6$.

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$\frac{1}{2}$ Why?

Assume for contradiction $aH \cap bH = \phi$. Since we are given $aH = bK$, It follows $bK \cap bH = \phi$, Concluding $H \cap K = \phi$. ? Contradiction as the identity element $e \in G$ is common in both subgroups. Therefore $aH \cap bH \neq \phi$.

$b \in bK$ Then $b \in aH$ but $aH \cap bH = \phi$ #

From *Lemma* (page 139), We get $aH = bH$. Then $bK = bH$ as we are given $aH = bK$. It follows $K = H$.

How?

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$\frac{0.5}{0.5}$

Let H be a proper subgroup of G. If $|H| = 1$, Then $H = \{e\} = \langle e\rangle$. it is cyclic. Now assume $|H| > 1$. Then by *theorem 7.1* (page 143), and without the loss of generality, $|H| = p$ for a prime p. By *corollary 3*, H is cyclic.

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$\frac{0.5}{1}$

$5^{16} \mod 7 = 6$ and $7^{13} \mod 11 = 2$, Using the fact $ab \mod m = (a \mod m)(b \mod m) \mod m$.

Use Fermat's last theorem

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Let H and K be finite subgroups of a group G, Where $|H|$ and $|K|$ are coprime. Since $H \cap K$ is a subgroup of both H and K, By *theorem 7.1* (page 142), $|H \cap K| = 1$. Then $H \cap K = \{e\}$, where e is the identity of G.

$\frac{0.75}{0.75}$

2
3

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We know all common divisors among 24 and 20 are $1, 2, 4$. By *theorem 7.1* (page 142), It follows $|H \cap K| = 1, 2$, or 4.

Case. $|H \cap K| = 1$. Then it is the trivial group of only the identity element.

Cyclic!

Case. $|H \cap K| = 2$. Then it is $\{e, a\}$. Trivially abelian.

Fact. For any two elements a, b of a group. If $ab = b$ then $a = e$, the identity element. Observe we can cancel b in $ab = eb = b$.

Case. $|H \cap K| = 4$. Assume for contradiction, that $ab \neq ba$ for arbitrary distinct elements a and b, Neither of which is the identity. Then $ab \notin \{a, b\}$ by the *Fact*. Moreover $ab \neq e$ lest $b = a^{-1}$ and then $ab = ba$. Symmetrically these conclusions apply on ba. Since we excluded 3 elements out of 4, There is only a single element ab and ba can both be assigned to, i.e $ab = ba$. Contradiction.

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