

$$\frac{5.75}{10}$$

Chapter 09

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Problems

1

We use *Theorem 9.1* (page 175) to show the answer is NO. $(23) \in S_3$ and yet, $(23)H(23) = \{(23)(1)(23), (23)(12)(23)\} = \{(1), (13)\} \notin H.$

$$\frac{0 \cdot S}{0 \cdot S}$$

2

We use *Theorem 9.1* (page 174). We know from earlier chapters, A_n is a subgroup of S_n . Then for any $x \in S_n$ and any $h \in A_n$, we get a permutation xhx^{-1} consisting of even 2-cycles. To see why, Observe we know x^{-1} has the same number of 2-cycles as x . Whether x consists of even or odd number of 2-cycles, The contribution of 2-cycles of both x and x^{-1} is even.

$$\frac{0 \cdot 7S}{0 \cdot 7S}$$

6

NO. It suffices to take some matrix $h \in H$ and a matrix $x \in GL(2, R)$, and show $xhx^{-1} \notin H$. Clearly:

$$\begin{aligned} & \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 4/3 & -1 \\ -1/3 & 2 \end{bmatrix} \notin H \end{aligned}$$

$$\frac{2}{2}$$

8

We immediately prove the general case of $\langle k \rangle / \langle n \rangle \cong \mathbb{Z}_{n/k}$, given k divides n .

For arbitrary two elements of under the operation:

$$\begin{aligned} (k^a \langle n \rangle)(k^b \langle n \rangle) &= k^{a+b} \langle n \rangle && \text{Definition} \\ &= k^{\frac{n}{k}q+r} \langle n \rangle, 0 \leq r < n/k && \text{Euclidean Division} \\ &= k^{\frac{n}{k}q} k^r \langle n \rangle \\ &= k^r (k^{\frac{n}{k}q} \langle n \rangle) && \text{Commutativity and Associativity of } \mathbb{Z} \\ &= k^r \langle n \rangle \\ &= k^r \langle n \rangle \end{aligned}$$

But in $\mathbb{Z}_{n/k}$, $ab = a + b \pmod{\frac{n}{k}}$, which corresponds to $(k^a \langle n \rangle)(k^b \langle n \rangle) = k^{a+b \pmod{\frac{n}{k}}} \langle n \rangle$.

$$\frac{0 \cdot 2S}{1}$$

1

Injectivity? Surjectivity?
2 Def. of the map?

9

Fact. Citing from the course TA, Ibrahim, left/right cosets partition the group G . (msh naseek ya bob). *Leul!*



Since the index is given to be 2, We know $G/H = \{H, g_0H\} = \{H, Hg_0\}$.

Consider arbitrary $x \in G$. If $x \in H$ then $xH = H = Hx$ from Lemma (page 139). If $x \notin H$, Then $x \in g_0H$ and $x \in Hg_0$ by our Fact. It follows $g_0h_0 = x = h_1g_0$ for some $h_0, h_1 \in H$, and in turn $xH = g_0H = Hg_0 = Hx$.

It follows H is normal.

As if you're using what you're trying to prove!
It's simpler than this!
 ~~G/H~~ Since $G = H \cup g_0H = H \cup Hg_0$
 with $g_0 \notin H$. & $H \cap g_0H = \emptyset = H \cap Hg_0$.
 Then $G/H = g_0H = Hg_0$.

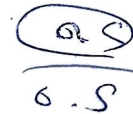
10

(a).

By Theorem 9.1 (page 175), We construct $xhx^{-1} \notin H$ for some $x \in A_4$ and $h \in H$.

Let $h = (12)(34)$ and $x = (13)(23)$. Then $x^{-1} = (23)(13)$, and in turn $xhx^{-1} = (13)(23)(12)(34)(23)(13)$. In other notation,

$$xhx^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix} \neq (12)(34)$$

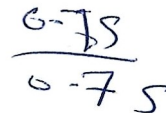


12

For arbitrary abelian group G with elements a_0 and a_1 , and factor group G/H , We have:

$$\begin{aligned} (a_0H)(a_1H) &= (a_0a_1)H \\ &= (a_1a_0)H \\ &= (a_1H)(a_0H) \end{aligned}$$

Definition
 G is Abelian



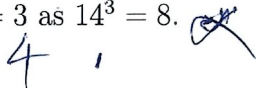
14

We know the identity of $\mathbb{Z}_{24}/\langle 8 \rangle$ is $0 + \langle 8 \rangle$. We are looking for smallest k satisfying

$$\begin{aligned} (14 + \langle 8 \rangle)^k &= 0 + \langle 8 \rangle \\ 14^k + \langle 8 \rangle &= \end{aligned}$$

Thanks for the course TA, Ibrahim, That can be satisfied while $14^k \neq 0$.

From the lemma (page 139), This is true if and only if $14^k \in \langle 8 \rangle$. In other words, We want smallest positive k , such that $14^k = 8^m$ for some integer m . By computation, $k = 3$ as $14^3 = 8 \cdot 14^2$.



3

22

Observe $(Z \oplus Z)/\langle(2, 2)\rangle = \{(0, 0) + \langle(2, 2)\rangle, (0, 1) + \langle(2, 2)\rangle, (1, 0) + \langle(2, 2)\rangle, (1, 1) + \langle(2, 2)\rangle\}$. To see why consider arbitrary $(a, b) \in Z \oplus Z$ and apply Euclid's division theorem to get $a = 2k_0 + r_0$ and $b = 2k_1 + r_1$ where $0 \leq r_0, r_1 < 2$.

Then the order is 4.

Infinite & not cyclic!

It is not cyclic as no single (a, b) can generate all of $(0, 0), (0, 1), (1, 0), (1, 1)$.

37

Recall the notation of $|g|$ as the order of element g . By definition $g^{|g|} = 0$. Then $(gH)^{|g|} = g^{|g|}H = H$. By Corollary 2 (page 77), $|gH|$ divides $|g|$.

①

$\frac{0}{2}$