# Chapter 09

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# Problems

## 1

We use *Theorem 9.1* (page 175) to show the answer is NO. $(23) \in S_3$ and yet, $(23)H(23) = \{(23)(1)(23), (23)(12)(23)\} = \{(1), (13)\} \not\subset H$.

## 2

We use *Theorem 9.1* (page 174). We know from earlier chapters, $A_n$ is a subgroup of $S_n$. Then for any $x \in S_n$ and any $h \in A_n$, we get a permutation $xhx^1$ consisting of even 2-cycles. To see why, Observe we know $x^{-1}$ has the same number of 2-cycles as $x$. Whether $x$ consists of even or odd number of 2-cycles, The contribution of 2-cycles of both $x$ and $x^{-1}$ is even.

## 6

NO. It suffices to take some matrix $h \in H$ and a matrix $x \in GL(2, R)$, and show $xhx^{-1} \notin H$. Clearly:

$$\begin{aligned}
&\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \\
&\quad = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 4/3 & -1 \\ -1/3 & 2 \end{bmatrix} \notin H
\end{aligned}$$

## 8

We immediately prove the general case of $\langle k \rangle / \langle n \rangle \cong \mathcal{Z}_{n/k}$, given $k$ divides $n$.

For arbitrary two elements of under the operation:

$$\begin{aligned}
(k^a\langle n \rangle)(k^b\langle n \rangle) &= k^{a+b}\langle n \rangle && \text{Definition} \\
&= k^{\frac{n}{k}q+r}\langle n \rangle, 0 \le r < n/k && \text{Euclidean Division} \\
&= k^{\frac{n}{k}q}k^r\langle n \rangle \\
&= k^r(k^{\frac{n}{k}q}\langle n \rangle) && \text{Commutativity and Associativity of } \mathcal{Z} \\
&= k^r(n\langle n \rangle) \\
&= k^r\langle n \rangle
\end{aligned}$$

But in $\mathcal{Z}_{n/k}$, $ab = a + b \mod \frac{n}{k}$, which corresponds to $(k^a\langle n \rangle)(k^b\langle n \rangle) = k^{a+b \mod r}\langle n \rangle$.

## 9

**Fact.** Citing from the course TA, Ibrahim, left/right cosets parition the group $G$. (msh naseek ya bob).

Since the index is given to be 2, We know $G/H = \{H, g_0H\} = \{H, Hg_0\}$.

Consider arbitrary $x \in G$. If $x \in H$ then $xH = H = Hx$ from *Lemma* (page 139). If $x \notin H$, Then $x \in g_0H$ and $x \in Hg_0$ by our *Fact*. It follows $g_0h_0 = x = h_1g_0$ for some $h_0, h_1 \in H$, and in turn $xH = g_0H = Hg_0 = Hx$.

It follows $H$ is normal.

## 10

### (a).

By *Theorem 9.1* (page 175), We construct $xhx^{-1} \notin H$ for some $x \in A_4$ and $h \in H$.

Let $h = (12)(34)$ and $x = (13)(23)$. Then $x^{-1} = (23)(13)$, and in turn $xhx^{-1} = (13)(23)(12)(34)(23)(13)$. In other notation,

$$xhx^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix} \neq (12)(34)$$

## 12

For arbitrary abelian group $G$ with elements $a_0$ and $a_1$, and factor group $G/H$, We have:

$$\begin{aligned} (a_0H)(a_1H) &= (a_0a_1)H && \text{Definition} \\ &= (a_1a_0)H && G \text{ is Abelian} \\ &= (a_1H)(a_0H) \end{aligned}$$

## 14

We know the identity of $\mathcal{Z}_{24}/\langle 8\rangle$ is $0 + \langle 8\rangle$. We are looking for smallest $k$ satisfying

$$\begin{aligned} (14 + \langle 8\rangle)^k &= 0 + \langle 8\rangle \\ 14^k + \langle 8\rangle &= \end{aligned}$$

Thanks for the course TA, Ibrahim, That can be satisfied while $14^k \neq 0$.

From the *lemma* (page 139), This is true if and only if $14^k \in \langle 8\rangle$. In other words, We want smallest positive $k$, such that $14^k = 8^m$ for some integer $m$. By computation, $k = 3$ as $14^3 = 8$.

## 22

Observe $(Z \oplus Z)/\langle(2,2)\rangle = \{(0,0) + \langle(2,2)\rangle, (0,1) + \langle(2,2)\rangle), (1,0) + \langle(2,2)\rangle, (1,1) + \langle(2,2)\rangle\}$. To see why consider arbitrary $(a,b) \in Z \oplus Z$ and apply Euclid's division theorem to get $a = 2k_0 + r_0$ and $b = 2k_1 + r_1$ where $0 \leq r_0, r_1 < 2$.

Then the order is 4.

It is not cyclic as no single $(a,b)$ can generate all of $(0,0), (0,1), (1,0), (1,1)$.

## 37

Recall the notation of $|g|$ as the order of element $g$. By definition $g^{|g|} = 0$. Then $(gH)^{|g|} = g^{|g|}H = H$. By *Corollary 2* (page 77), $|gH|$ divides $|g|$.