Chapter 13

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Problems

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Let R be a commutative ring with the cancellation property. Assume for contradiction a is a zero-divisor. Then $a \neq 0$ and there's $b \neq 0$ such that $ab = 0$. By *theorem 12.1* (page 229) It follows:

$$\begin{aligned} ab - ab &= 0 \\ a(b-b) &= 0 \\ &= 0 \cdot 0 \\ &= 0 \cdot (b-b) \\ a &= 0 \end{aligned}$$

Contradiction.

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Zero-divisors are $2, 4, 6, 8, 10, 12, 14, 15, 16, 18, 5, 15$. To see why assume $ab \mod 20 \equiv 0 \mod 20$. Then $ab - 0 = ab = 20k$ for some k. a must contain a common factor with 20, as otherwise $b \geq 20$. So zero-divisors are multiples of a factor of 20.

Unities are $1, 3, 7, 9, 11, 13, 17, 19$. No proof is found.

Zero-divisors and unities partition $\mathbb{Z}_{20}$.

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We call $a \in Z \oplus Z \oplus Z$ *strictZero* if some component of a is 0 like $(x, y, 0)$ but $a \neq (0, 0, 0)$.

Clearly a is a zero-divisor if and only if a is a *strictZero*.

For $a, b, c \in Z \oplus Z \oplus Z$, if ab, ac, bc are zero-divisors then they are *strictZeros*. If abc is not a zero-divisor then it is not a *strictZero*, in other words either $abc = (0, 0, 0)$, or $abc = (x, y, z)$ where $x, y, z \neq 0$. The latter case cannot happen as ab is a a *strictZero* so some component must be zero in abc. Therefore $abc = (0, 0, 0)$.

Since Z has no zero-divisor, it follows each component is zero in one of a, b, c. In other words, The a, b, c we are characterizing, are *strictZeros*, such that no component is non-zero in the three of them.

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Let R be an integral domain and $a^2 = a$. Then

$$\begin{aligned} a^2 - a &= 0 \\ a(a-1) &= 0 \end{aligned}$$

Since there are no zero-divisors, either $a = 0$ or $a - 1 = 0$.

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We know R is a group. By usual properties of addition and multiplication, It is a commutative ring.

6 is unity as

$$6 \cdot 0 = 0$$
$$6 \cdot 2 = 2$$
$$6 \cdot 4 = 4$$
$$6 \cdot 6 = 6$$
$$6 \cdot 8 = 8$$

Each non-zero element has a unit as

$$2 \cdot 8 = 6$$
$$4 \cdot 4 = 6$$
$$6 \cdot 6 = 6$$
$$8 \cdot 2 = 6$$

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Observe by distributivity of rings $x^2 - 5x + 6 = (x - 3)(x - 2)$.

a

By the *corollary* (page 239) $\mathbb{Z}_7$ is a field, and hence has no zero-divisors. It follows either $x - 3 = 0$ or $x - 2 = 0$ so $x = 3$ or $x = 2$. Exactly two solutions.

b

By computation 2 and 3 are the solutions.

Note nothing certifies $\mathbb{Z}_8$ is an integral domain.