Ex1

Considering only one binary Connector $\diamond$

define

$$x_0 = |F_0|$$
$$x_n = |F_n \setminus F_{n-1}|$$

Remark. a formula is in $F_n \setminus F_{n-1}$ iff it has a subformula in $F_{n-1} \setminus F_{n-2}$

Then

$$x_n = (x_{n-1} \cdot |F_{n-2}| \cdot 2) + (x_{n-1})^2$$
$$= (x_{n-1} \cdot |F_{n-1}| \cdot 2) - (x_{n-1})^2$$

$$|F_n| = |F_{n-1}| + x_n$$

Proof 1.

$$\{\psi \mid \psi \in F_n \setminus F_{n-1}\} = \{\psi \mid \psi = (\lambda \diamond \theta), \text{ Exactly } \psi \text{ or } \lambda \text{ is in } F_{n-1} \setminus F_{n-2}\}$$
$$\cup$$
$$\{\psi \mid \psi = (\lambda \diamond \theta), \psi \in F_n \setminus F_{n-1}, \lambda, \theta \in F_{n-1} \setminus F_{n-2}\}$$

On R.H.S, The second set has a bijection with the cartesian product $|F_{n-1} \setminus F_{n-2}| \times |F_{n-1} \setminus F_{n-2}|$, So its count is $(x_{n-1})^2$

On R.H.S, The first set has two distinct elements $(\lambda_0 \diamond \theta_0)$ and $(\theta_0 \diamond \lambda_0)$ sharing the same subformulas. So we multiply by 2, On the Cartesian product $|F_{n-1} \setminus F_{n-2}| \times |F_{n-2}|$

The intersection of sets on R.H.S is the empty set, So we take the summation

Proof 2.

Lemma. Cardinality of $C = \{ \tau \mid \tau = (\lambda \diamond \theta), \tau \in F_n \setminus F_{n-1}, \lambda, \theta \in F_{n-1} \setminus F_{n-2} \}$ is $(\chi_{n-1})^2$

Cartesian Product

define a function $f : C \to F_{n-1} \setminus F_{n-2} \times F_{n-1} \setminus F_{n-2}$, where $f(\tau) = (\lambda, \theta)$ iff $\tau = (\lambda \diamond \theta)$

it's injective, as if $f(\tau_0) = f(\tau_1)$, Then $\tau_0 = (\lambda, \theta) = \tau_1$
it's surjective, as for each $\lambda, \theta \in F_{n-1} \setminus F_{n-2}$, We can Construct $\tau = (\lambda \diamond \theta)$ whereby $f(\tau) = (\lambda, \theta)$.
Hence f is bijective.

But the cardinality of the co-domain is $|F_{n-1} \setminus F_{n-2}| \times |F_{n-1} \setminus F_{n-2}|$, and so is the cardinality of the domain.