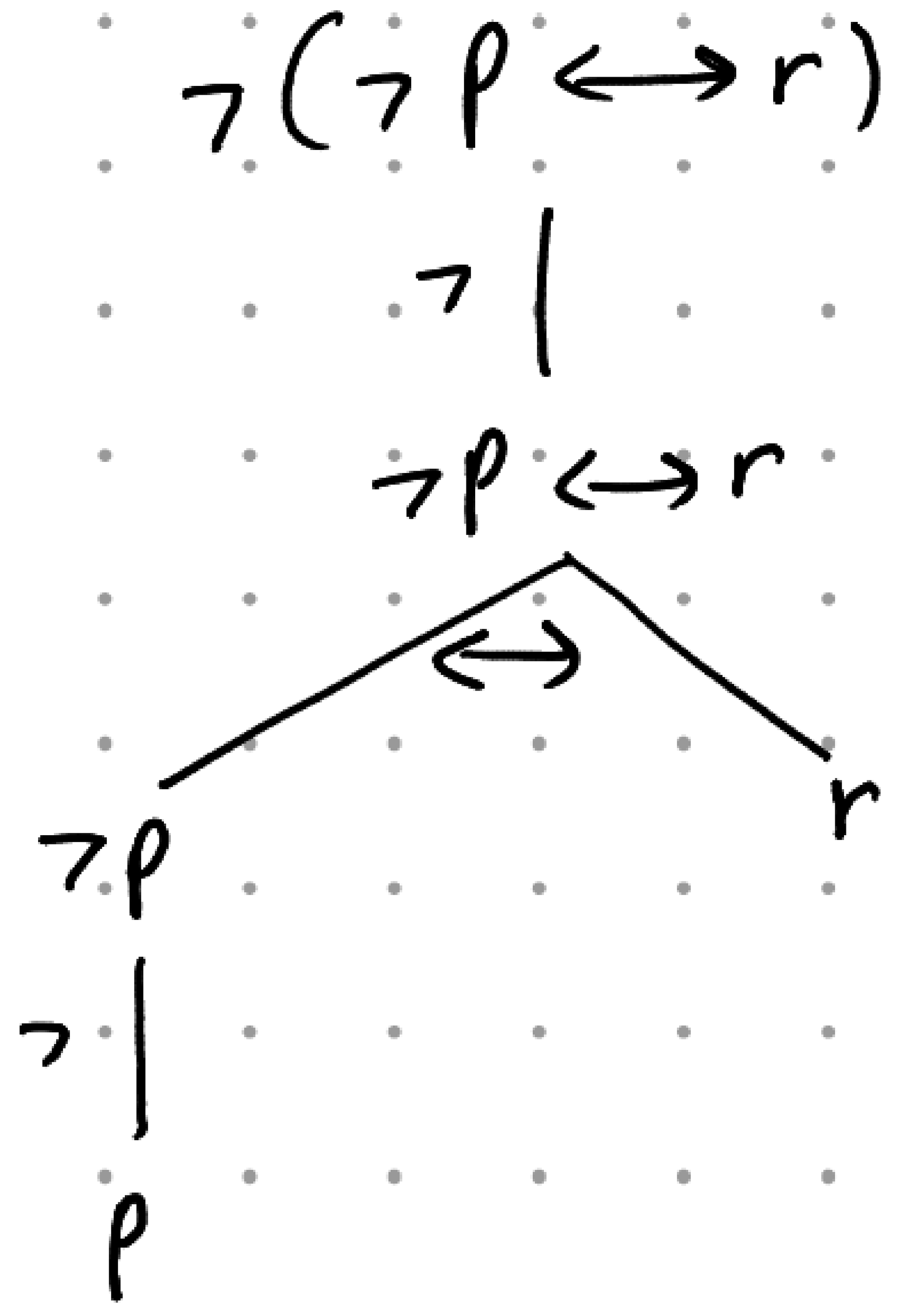
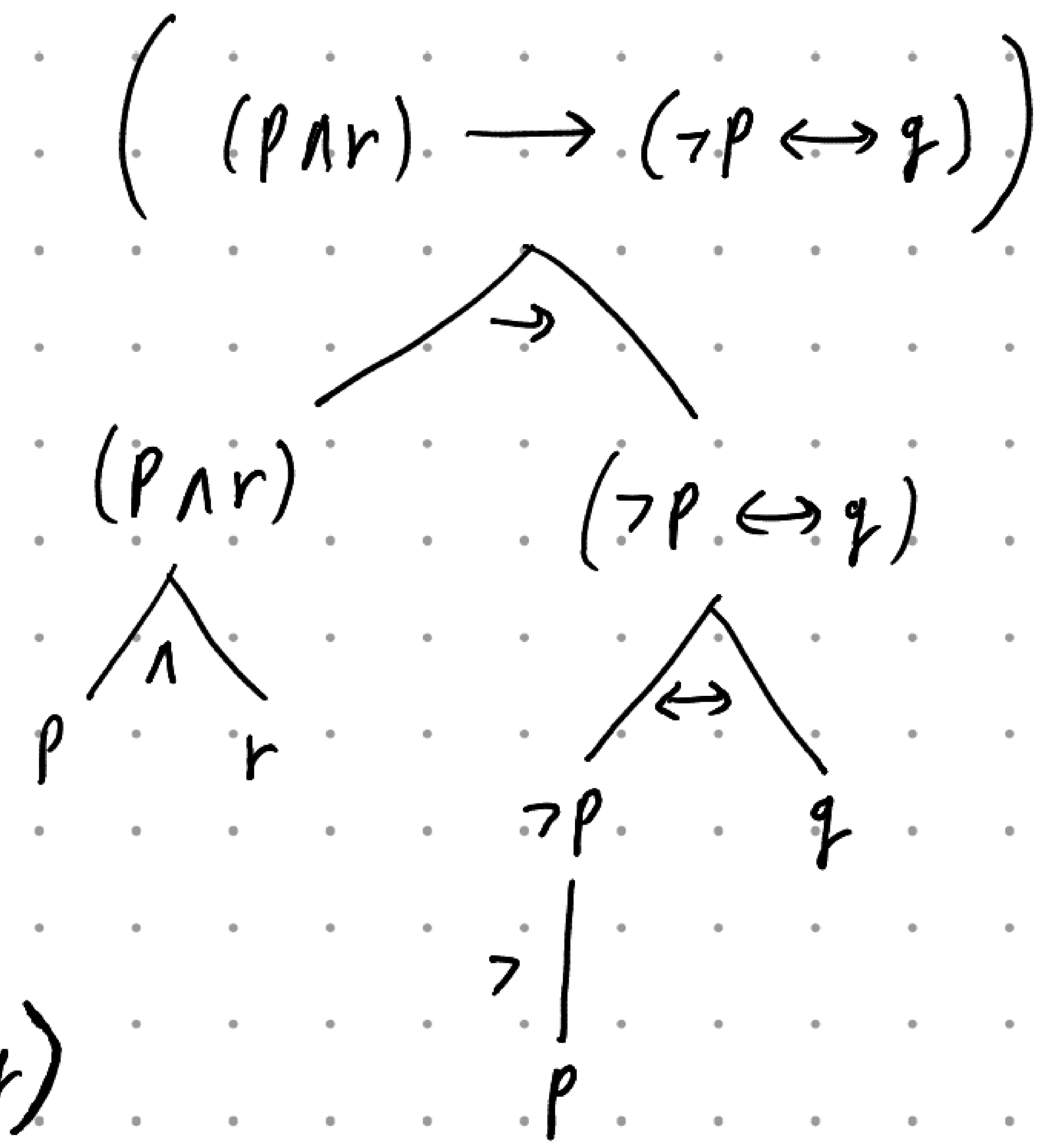
Ex. 1

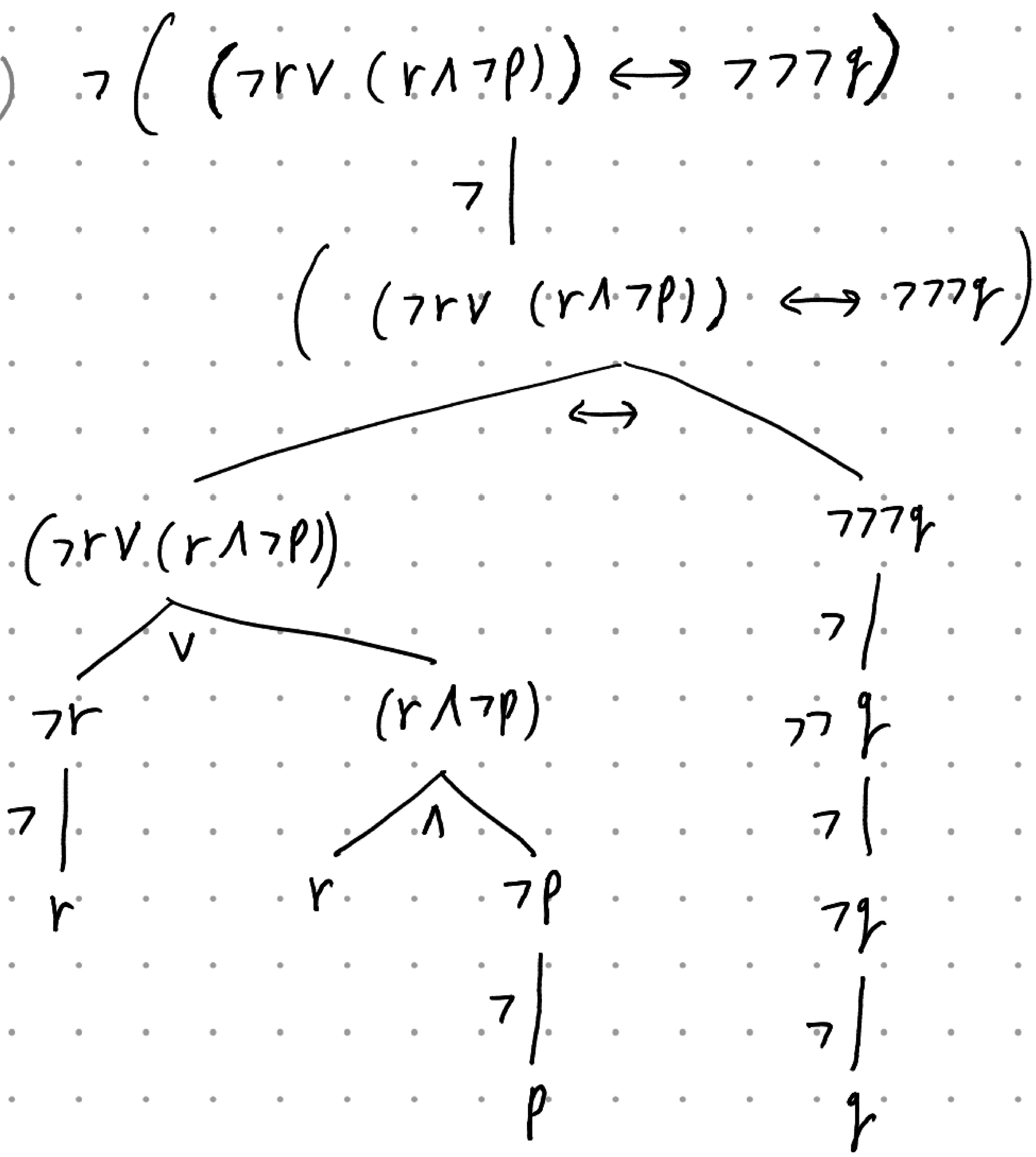
1)



2)



3)



all nodes are sub-formulas. follows from the def. of subformulas

Ex. 2

Base case. Propositional variables

$\#\wedge[\psi] = 0 = \#)[\psi]$. holds

induction hypo. holds for $\psi \in F_n$

induction step. $\psi \in F_{n+1}$

by decomposition

Case. $\psi = \neg\theta$, $\theta \in F_n$. Then $\#\wedge[\theta] \leqslant \#)[\theta]$

clearly $\#\wedge[\neg\theta] = \#\wedge[\theta]$

$\#)[\neg\theta] = \#)[\theta]$

Concluding

$\#\wedge[\neg\theta] \leqslant \#)[\neg\theta]$

$\#\wedge[\psi] \leqslant \#)[\psi]$

Case. $\psi = (\theta \lozenge \lambda)$, $\theta, \lambda \in F_n$. Then $\#\wedge[\theta] \leqslant \#)[\theta]$

$\#\wedge[\lambda] \leqslant \#)[\lambda]$

$$
\begin{aligned}
\#[\psi] &= \#\wedge[(] + \#\wedge[\theta] + \#\wedge[\lozenge] + \#\wedge[\lambda] + \#\wedge[)] \\
&= 0 + \#\wedge[\theta] + 1 + \#\wedge[\lambda] + 0 \\
&\leqslant \#)[\theta] + \#)[\lambda] + 1 \\
&= 0 + \#)[\theta] + 0 + \#)[\lambda] + 1 \\
&= \#)[(] + \#)[\theta] + \#)[\lozenge] + \#)[\lambda] + \#)[)] \\
&= \#)[\psi]
\end{aligned}
$$

further. Rigorous proof of

$\#\neg[\neg w] = \#\neg[w] + 1$

$\#\neg[cw] = \#\neg[w]$, $c \neq \neg$

Ex. 3

1) Following the def at p.15,

a) $\delta[\neg q] = \delta[q]+1 = 0+1 = 1$

b) $\delta[\neg p] = \delta[p]+1 = 1+1 = 0$

$\delta[(\neg p \vee r)] = \delta[\neg p] + \delta[r] + \delta[\neg p]\cdot\delta[r]$

$= 0 + 0 + 0\cdot 0 = 0$

c) $\delta[\neg r] = \delta[r]+1 = 0+1 = 1$

$\delta[r \wedge \neg p] = \delta[r]\cdot\delta[\neg p] = 0\cdot 0 = 0$

$\delta[(\neg r \vee (r\wedge\neg p))] = \delta[\neg r] + \delta[(r\wedge\neg p)] + \delta[\neg r]\cdot\delta[(r\wedge\neg p)]$

$= 1 + 0 + 1\cdot 0 = 1$

$\delta[\neg\neg q] = \delta[\neg q]+1 = 1+1 = 0$

$\delta[\neg\neg\neg q] = \delta[\neg\neg q]+1 = 0+1 = 1$

$\delta[\ (\neg r \vee (r\wedge\neg p)) \leftrightarrow \neg\neg\neg q\ ]$

$= 1 + \delta[(\neg r \vee (r\wedge\neg p))] + \delta[\neg\neg\neg q]$

$= 1 + 1 + 1 = 1$

$\delta[\neg(\ (\neg r\vee(r\wedge\neg p)) \leftrightarrow \neg\neg\neg q\ )]$

$= 1+1 = 0$

2) $\psi(p, \theta/r) = \Big(p \leftrightarrow (\neg(\neg p \vee r) \rightarrow p)\Big)$

$\psi(\psi/p, \theta/r) = \Big((p \leftrightarrow (\neg r \rightarrow p)) \leftrightarrow (\neg(\neg p \vee r) \rightarrow (p \leftrightarrow (\neg r \rightarrow p)))\Big)$

Ex. 4

let $\varphi \in F$

Then $h[\varphi] = n$ for some $n \geq 0$

So $\varphi \in F_n \setminus F_{n-1}$

by def $\neg\varphi \in F_{n+1}$

Assume for contradiction $\neg\varphi \in F_K$ for any $K < n+1$

by decomposition, $\varphi \in F_{K'}$, where $K' < K < n+1$. So $K' < n$

Contradicting the fact $h[\varphi] = n$

Ex. 5

$\phi \equiv \psi$

$\leftrightarrow$ $\delta[\phi] = \delta[\psi]$, for any truth assignment δ, by def of 1.2.2

$\leftrightarrow$ $\delta[\phi \leftrightarrow \psi] = 1$, for any truth assignt δ, by thm 1.2.1

Ex. 6

1) already solved in notes

2) 2^{2^n}. it follows by the observation that a $\equiv$-equivalence class is decided by the binary truth-value assignments of the 2^n tuples $\{0,1\}^n$.