Ex. 1

1. Equivalent.

   let δ be an arbitrary truth assignment. By def

$$\delta[\neg(p\wedge r)] = 1+\delta[(p\wedge r)]$$
$$= 1+\delta[p]\cdot\delta[r]$$

$$\delta[(\neg p\vee\neg r)] = \delta[\neg p]+\delta[\neg r]+\delta[\neg p]\delta[\neg r]$$
$$= 1+\delta[p]+1+\delta[r]+(1+\delta[p])(1+\delta[r])$$
$$= \delta[p]+\delta[r]+1+\delta[p]+\delta[r]+\delta[p]\delta[r]$$
$$= 1+\delta[p]\delta[r]$$

2. Equivalent.

   let δ be an arbitrary truth assignment. By def.

$$\delta[((p\wedge q)\wedge r)] = \delta[(p\wedge q)]\cdot\delta[r]$$
$$= \delta[p]\cdot\delta[q]\cdot\delta[r]$$

$$\delta[(p\wedge(q\wedge r))] = \delta[p]\cdot\delta[(q\wedge r)]$$
$$= \delta[p]\cdot\delta[q]\cdot\delta[r]$$

3. Equivalent.

   let δ be an arbitrary truth assignment. By def.

$$\delta[(p\vee q)\vee r] = \delta[(p\vee q)]+\delta[r]+\delta[(p\vee q)]\delta[r]$$
$$= \delta[p]+\delta[q]+\delta[p]\delta[q]+\delta[r]+(\delta[p]+\delta[q]+\delta[p]\delta[q])\delta[r]$$
$$= \delta[p]+\delta[q]+\delta[p]\delta[q]+\delta[r]+\delta[p]\delta[r]+\delta[q]\delta[r]+\delta[p]\delta[q]\delta[r]$$

$$\delta[(p\vee(q\vee r))] = \delta[p]+\delta[(q\vee r)]+\delta[p]\delta[(q\vee r)]$$
$$= \delta[p]+\delta[q]+\delta[r]+\delta[q]\delta[r]+\delta[p](\delta[q]+\delta[r]+\delta[q]\delta[r])$$
$$= \delta[p]+\delta[q]+\delta[r]+\delta[q]\delta[r]+\delta[p]\delta[q]+\delta[p]\delta[r]+\delta[p]\delta[q]\delta[r]$$

4. Not Equivalent

Consider truth assignment δ_0 where $\delta_0[p] = \delta_0[r] = 0$

$$\delta[((p\to q)\to r)] = 1 + \delta[(p\to q)] + \delta[(p\to q)]\delta[r]$$
$$= 1 + (1 + \delta[p] + \delta[p]\delta[q]) + (1 + \delta[p] + \delta[p]\delta[q])\delta[r]$$
$$= 1 + 1 + 0$$
$$= 0$$

$$\delta[(p\to(q\to r))] = 1 + \delta[p] + \delta[p]\delta[(q\to r)]$$
$$= 1 + \delta[p] + \delta[p](1 + \delta[q] + \delta[q]\delta[r])$$
$$= 1 + 0 + 0$$
$$= 1$$

Ex. 2

Lemma L1. $\neg\neg p \equiv p$. clearly $\delta[\neg\neg p] = 1 + \delta[\neg p] = 1 + 1 + \delta[p] = \delta[p]$

1. Tautology. $(p \vee \neg p)$ is in DNF

let δ be any truth-assignment

$$\delta[p\vee\neg p] = \delta[p] + \delta[\neg p] + \delta[p]\cdot\delta[\neg p] = \delta[p] + (1+\delta[p]) + \delta[p](1+\delta[p])$$
$$= 1 + \delta[p] + \delta[p]\delta[p] = 1 + \delta[p] + \delta[p] = 1$$

2. tautology. DNF: $(q \vee p \vee \neg p)$. By lemma 1.2.9 and Corollary 1.2.7

$$(((p\to q)\wedge\neg q)\to\neg p)$$

$$(((\neg p \vee q)\wedge\neg q)\to\neg p) \qquad (18)$$

$$(\neg((\neg p\vee q)\wedge\neg q) \vee \neg p) \qquad (18)$$

$((\neg(\neg p \vee q) \vee \neg\neg q) \vee \neg p)$ (11)

$(((\neg\neg p \wedge \neg q) \vee q) \vee \neg p)$ (12)

$(((p \wedge \neg q) \vee q) \vee \neg p)$ (L1)

$(((p \vee q) \wedge (\neg q \vee q)) \vee \neg p)$ (8)

$(((p \vee q) \wedge T) \vee \neg p)$ lemma 1.2.10 (1)

$((p \vee q) \vee \neg p)$ (13)

$((q \vee p) \vee \neg p)$ (4)

$(q \vee (p \vee \neg p))$ (6)

$(q \vee T)$ lemma 1.2.10(1)

T (16)

3. Tautology. DNF : $\neg p \vee p$

$((\neg p \to p) \leftrightarrow p)$

$((p \vee p) \leftrightarrow p)$ (18)

$(p \leftrightarrow p)$ (2)

$(p \to p) \wedge (p \to p)$ (21)

$p \to p$ (1)

$\neg p \vee p$ (18)

T lemma 1.2.10 (1)

4. Tedious

Ex. 3

1. Fact. $\delta^*[p] = 1 + \delta[p]$ and $\delta[p] = 1 + \delta^*[p]$

Fact. $[\neg\phi]^* = \neg\phi^*$ and $[(\phi \lozenge \lambda)]^* = (\phi^* \lozenge \lambda^*)$

Approach. Induction on formulas.

Base Case. F_0. $\phi = p$ for some $p \in P$

Then $\phi^* = \neg p$. It follows

$$\delta[p] = \delta[p] + 1 + 1$$
$$\delta[\phi] = \delta^*[p] + 1$$
$$= \delta^*[\neg p]$$
$$= \delta^*[\phi^*]$$

Induction hypothesis. Assume the statement holds for $n \geq 0$, on formulas F_n.

Induction Step. $\psi \in F_{n+1}$. Then

Case 1. $\psi = \neg\phi$, where $\phi \in F_n$

clearly $\psi^* = [\neg\phi]^* = \neg\phi^*$

by ind. hypo. $\delta[\phi] = \delta^*[\phi^*]$. Then

$$1 + \delta[\phi] = 1 + \delta^*[\phi^*]$$
$$\delta[\neg\phi] = \delta^*[\neg\phi^*]$$
$$\delta[\psi] = \delta^*[\psi^*]$$

Case 2. $\psi = (\phi \lozenge \lambda)$, where $\phi, \lambda \in F_n$

clearely $\psi^* = [(\phi \lozenge \lambda)]^* = (\phi^* \lozenge \lambda^*)$

by ind. hypo. $\delta[\phi] = \delta^*[\phi^*]$ and $\delta[\lambda] = \delta^*[\lambda^*]$

Case 2.1. $\lozenge = \wedge$

$$\delta[\psi] = [\phi \wedge \lambda] = \delta[\phi] \cdot \delta[\lambda]$$
$$= \delta^*[\phi^*] \delta^*[\lambda^*]$$
$$= \delta^*[\psi^*]$$

Case 2.2 $\lozenge = \rightarrow$

$$\delta[\psi] = 1 + \delta[\phi] + \delta[\phi] \cdot \delta[\lambda]$$
$$= 1 + \delta^*[\phi^*] + \delta^*[\phi^*] \delta^*[\lambda^*]$$
$$= \delta^*[\psi^*]$$

2. Observe $\phi^* = \phi(\neg p_1/p_1, \neg p_2/p_2, \dots, \neg p_n/p_n)$

Consider truth-assignment λ where $\lambda[p_i] = \delta[\neg p_i] = 1 + \delta[p_i]$

($\rightarrow$) Given ϕ is a tautology

$$\delta[\phi(\neg p_1/p_1, \dots, \neg p_n/p_n)] = \lambda[\phi] \quad \text{by thm 1.2.5}$$
$$\delta[\phi^*] = 1 \quad \text{since } \phi \text{ is a tautology}$$

Since δ was arbitrary, ϕ^* is a tautology.

($\leftarrow$) Given ϕ^* is a tautology

Then $[\phi^*]^*$ is a tautology as we have just proved

but $\neg\neg p_i \equiv p_i$, and hence by Corollary 1.2.7 $\phi^{**} \equiv \phi$

So ϕ is a tautology.

alt. Proof using (1)

($\rightarrow$) Given ϕ is tautology

let δ be any truth-assignment.

Construct δ^* as given in the question. Then

$$\begin{aligned} \delta(\phi^*) &= \delta^*(\phi^{**}) && \text{By (1)} \\ &= \delta^*(\phi) && \phi \text{ and } \phi^{**} \text{ are logically equivalent} \\ &= 1 && \phi \text{ tautolgy} \end{aligned}$$

($\leftarrow$) Symmetric

alt Proof (By TA Ibrahim)

We have $T: S \to S$, $S = \{\text{truth-assignments}\}$

$\delta \mapsto \delta^*$, s.t $\delta(\phi) = \delta^*(\phi^*)$

We show T is surjective.

for $\delta \in S$, Consider δ^*, and observe $T(\delta^*) = \delta^{**} = \delta$.

Now taking arbitrary δ, We know there's δ' s.t $\delta'(\phi) = \delta(\phi^*)$

but ϕ is given tautology, So $\delta'(\phi) = 1$