Mathematical Logic
Assignment 4

Exercise 1 **(5+10+10)**

Prove that all instances of the axioms of Hilbert system $\mathcal{S}$ are tautologies.

Exercise 2 **(10+10)**

Let ϕ and ψ be formulas. Prove that the following hold

1. $(\phi \equiv \psi)$ if and only if $\phi \models \psi$ and $\psi \models \phi$.
2. $\Gamma \cup \{\phi\} \models \psi$ if and only if $\Gamma \models (\phi \to \psi)$.

Exercise 3 **(5+5)**

Prove the following statements:

1. Let $\top$ be any tautology and $\phi \in \mathcal{F}$, then $(\phi \to \top)$ is also a tautology.
2. Let $\top$ be any theory of the system $\mathcal{S}$ and $\phi \in \mathcal{F}^*$, then $\vdash (\phi \to \top)$.

Exercise 4 **(15)**

Prove that if $\Delta \vdash \varphi$ and $\Gamma \cup \{\varphi\} \vdash \psi$, then $\Delta \cup \Gamma \vdash \psi$.

Exercise 5 **(5+5+5+5)**

Prove the following using our Hilbert-style System $\mathcal{S}$. You are permitted to use the deduction theorem and theories of the system $\mathcal{S}$ that are proven in the notes. (hint: use 3 in 4)

1. $\vdash (\varphi \to ((\varphi \to \psi) \to \psi))$.
2. $\{(\varphi \to \psi), (\psi \to \theta)\} \vdash (\varphi \to \theta)$.
3. $\vdash (\neg\varphi \to (\varphi \to \psi))$ (Duns Scotus Law)
4. $\vdash ((\neg\varphi \to \varphi) \to \varphi)$ (Law of Clavius).