Fact (clear)

1. if $\mu \vdash \varphi$ then $\mu \cup \mu' \vdash \varphi$
2. if $\mu \vdash \varphi$ for all $\varphi \in \mu'$ and $\mu' \vdash \phi$, then $\mu \vdash \phi$

Ex 1

Ax1. $\left(\varphi \to (\psi \to \varphi)\right)$

let δ be arbitrary truth-assignment. Then by def

$$
\begin{aligned}
\delta[\varphi \to (\psi \to \varphi)] &= 1 + \delta[\varphi] + \delta[\varphi]\cdot\delta[\psi \to \varphi] \\
&= 1 + \delta[\varphi] + \delta[\varphi]\cdot\left(1 + \delta[\psi] + \delta[\psi]\delta[\varphi]\right) \\
&= 1 + \cancel{\delta[\varphi]} + \cancel{\delta[\varphi]} + \delta[\varphi]\delta[\psi] + \delta[\varphi]^2\delta[\psi] \\
&= 1 + \delta[\varphi]\delta[\psi] + \delta[\varphi]\delta[\psi] \\
&= 1
\end{aligned}
$$

Ax 3. $\left(\left(\neg\varphi \to \neg\psi\right) \to (\psi \to \varphi)\right)$

For any δ, $\delta\left[\left((\neg\varphi \to \neg\psi) \to (\psi \to \varphi)\right)\right]$

$$
\begin{aligned}
&= 1 + \delta[\neg\varphi \to \neg\psi] + \delta[\neg\varphi \to \neg\psi]\,\delta[\psi \to \varphi] \\
&= \cancel{1} + \left(\cancel{1} + \delta[\neg\varphi] + \delta[\neg\varphi]\delta[\neg\psi]\right) + \left(1 + \delta[\neg\varphi] + \delta[\neg\varphi]\delta[\neg\psi]\right)\left(1 + \delta[\psi] + \delta[\psi]\delta[\varphi]\right) \\
&= (1 + \delta[\varphi]) + (1 + \delta[\varphi])(1 + \delta[\psi]) + \left(\cancel{\delta[\varphi]} + 1 + \cancel{\delta[\varphi]} + \delta[\psi] + \delta[\varphi]\delta[\psi]\right) \\
&\qquad \left(1 + \delta[\psi] + \delta[\psi]\delta[\varphi]\right) \\
&= \cancel{1} + \cancel{\delta[\varphi]} + \cancel{1} + \cancel{\delta[\varphi]} + \cancel{\delta[\psi]} + \cancel{\delta[\varphi]\delta[\psi]} + 1 + \cancel{\delta[\psi]} + \cancel{\delta[\psi]\delta[\varphi]} \\
&\qquad + \cancel{\delta[\psi]} + \cancel{\delta[\psi]^2} + \cancel{\delta[\psi]^2\delta[\varphi]} \\
&\qquad + \cancel{\delta[\varphi]\delta[\psi]} + \cancel{\delta[\varphi]\delta[\psi]^2} + \cancel{\delta[\varphi]^2\delta[\psi]^2} \\
&= 1
\end{aligned}
$$

Ax 2.

lazy to typeset.

Ex. 2

1) ($\rightarrow$) Trivial.

($\leftarrow$) prove by Contrapositive. For any δ, if $\delta[\phi] \neq \delta[\psi]$, Then either

① $\delta[\phi]=1$ and $\delta[\psi]=0$, Concluding $\phi \not\models \psi$, or

② $\delta[\phi]=0$ and $\delta[\psi]=1$, Concluding $\psi \not\models \phi$

2) ($\rightarrow$) For any δ, Assume δ satisfies $\mathcal{M}$. Then $\delta[\phi\rightarrow\psi] = 1+\delta[\phi]+\delta[\phi]\delta[\psi]$. Either

① $\delta[\phi]=0$, Concluding $\delta[\phi\rightarrow] = 1+0+0=1$, or

② $\delta[\phi]=1$, Concluding δ satisfies $\mathcal{M}\cup\{\phi\}$, and by hypothesis $\delta[\psi]=1$. it follows $\delta[\phi\rightarrow\psi]=1+1+1=1$

($\leftarrow$) for any δ, Assume δ satisfies $\mathcal{M}\cup\{\phi\}$, Then δ satisfies also $\mathcal{M}$. By hypothesis $\delta[\phi\rightarrow\psi]=1$. Observe

$$\begin{aligned}\delta[\phi\rightarrow\psi] &= 1+\delta[\phi]+\delta[\phi]\delta[\psi]\\ 1 &= 1+1+\delta[\psi]\\ &= \delta[\psi]\end{aligned}$$

Ex. 3

1) let δ be arbitrary.

Case ①. $\delta[\phi]=0$, Then $\delta[\phi\rightarrow T]=1+0+0=1$

Case ②. $\delta[\phi]=1$, Then $\delta[\phi\rightarrow T]=1+1+1=1$

2) Since $\vdash T$, $\{\phi\}\vdash T$ by fact 1.

By the deduction theorem, $\vdash(\phi\rightarrow T)$

Ex 4

By the deduction theorem $\mathcal{M} \cup \{\varphi\} \vdash \psi$ implies $\mathcal{M} \vdash (\varphi \to \psi)$

Clearly $\Delta \cup \mathcal{M} \vdash \varphi$

$\Delta \cup \mathcal{M} \vdash \varphi \to \psi$ by fact 1

But $\{\varphi, \varphi \to \psi\} \vdash \psi$ using MP
it follows $\Delta \cup \mathcal{M} \vdash \psi$ by fact 2

Ex. 5

deduction
Contrad I
Contrad II

1) $\{\varphi\} \cup \{\varphi \to \psi\} \vdash \psi$ MP

$\{\varphi\} \vdash (\varphi \to \psi) \to \psi$ deduction

$\vdash \left(\varphi \to ((\varphi \to \psi) \to \psi)\right)$ deduction

2) $\{(\varphi \to \psi), (\psi \to \emptyset)\} \cup \{\varphi\} \vdash \emptyset$ MP

$\{(\varphi \to \psi), (\psi \to \emptyset)\} \vdash (\varphi \to \emptyset)$ deduction

3) That's exactly the first example in page 39 of the notes.

4) $\{(\neg\varphi \to \varphi)\} \cup \{\neg\varphi\} \vdash \varphi$ MP

$\{(\neg\varphi \to \varphi)\} \cup \{\neg\varphi\} \vdash \neg\varphi$

$\{(\neg\varphi \to \varphi)\} \vdash \varphi$ Contradiction

$\vdash \left((\neg\varphi \to \varphi) \to \varphi\right)$ deduction lemma. if $M \cup \{T\} \vdash \varphi$ and $\vdash T$, Then $M \vdash \varphi$

let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the derivation of $M \cup \{T\} \vdash \varphi$, and $\beta_1, \beta_2, \cdots, \beta_n$ be the derivation of $\vdash T$

Consider $\beta_1, \beta_2, \cdots, \beta_n, \alpha_1, \alpha_2, \cdots, \alpha_n$. Then T in $\{\alpha_i\}$ shall no longer be an assumption. in other words, it's a derivation from M. Since $\beta_n = \varphi$, clearly $M \vdash \varphi$.

Fact 2 Proof

let $\mu' = \gamma_1, \gamma_2, \gamma_3, \ldots$

We're given $\mu \vdash \gamma_i$. denote its derivation

$\varphi_{i,1}$

$\varphi_{i,2}$

$\varphi_{i,3}$

$\vdots$

$\varphi_{i,n_i} = \gamma_i$

We're given also from μ'

β_1

β_2

$\vdots$

$\beta_n = \emptyset$

Construct a sequence

$\varphi_{i,j}$

β_1

β_2

$\vdots$

$\beta_n = \emptyset$

Clearly this sequence relies on assumptions only from μ, as $\beta_i \in \mu'$ are now deduced from μ.

Thus $\mu \vdash \emptyset$