Facts (clear)

1. $\emptyset \cup \mu \vdash \emptyset$
2. if $\mu \models \emptyset$ then $\mu \cup \mu' \models \emptyset$

Ex. 1

1) Derivable.

$\neg(q \to \neg p) \to \big((\neg q \to p) \to \neg(q \to \neg p)\big)$	A1
$\neg(q \to \neg p)$	Assumption
$\big((\neg q \to p) \to \neg(q \to p)\big)$	MP

Concluding $\neg(q \to \neg p) \vdash \big((\neg q \to p) \to \neg(q \to \neg p)\big)$

2) Not derivable. Consider δ_0 where $\delta_0[p] = \delta_0[r] = 0$ and $\delta_0[q] = 1$. Then $\not\models\big((p \to (p \to \neg r)) \to (q \to r)\big)$. By Completeness the intended result follows.

Ex. 2

1) Not consistent. We use corallary 2.4.4, and show the set is not satisfiable.

let δ be an arbitrary truth-assignment. Then $\delta[\neg(p \to q)] = \delta[\neg(q \to r)] = 1$. So $\delta[(p \to q)] = \delta[(q \to r)] = 1 + 1 = 0$. By theorem 1.2.1, $\delta[q] = 0$ and $\delta[q] = 1$. Contradiction

2) Consistent. Using corallary 2.4.7, it suffices to show the given set is satisfiable.

Consider the truth-assignment δ_0 whereby $\delta_0[p] = 0$, $\delta_0[q] = \delta_0[r] = 1$ Then $\delta[(p \to q)] = \delta_0[(q \to r)] = \delta_0[(r \to \neg p)] = 1$, Hence satisfiable.

3) Not consistent. We use corallary 2.4.4, and show the set is not satisfiable.

let δ be an arbitrary truth-assignment. Then $\delta[\neg(p \to q)] = \delta[q] = 1$. $\delta[(p \to q)] = 0$. By theorem 1.2.1, $\delta[q] = 0$. Contradiction.

Ex. 3

$(\rightarrow)$ Trivial as $\mathcal{M} \vdash \varphi$ for any $\varphi \in \mathcal{M}$ by one line assumption.

($\Leftarrow$) Given $\Sigma \vdash \varphi$, there's a derivation sequence

$$\left.\begin{array}{l}\psi_0 \\ \psi_1 \\ \psi_2 \\ \vdots\end{array}\right] \begin{array}{l}\text{Assumptions from } \Sigma \\ \text{Axioms} \\ \text{MP}\end{array}$$

$\psi_n = \varphi$

let the assumptions be $\phi_1, \phi_2, \dots, \phi_m$. By definition we know $\mathcal{M} \vdash \phi_i$. Then we have for each ϕ_i, a derivation sequence (ϕ_{ij}).

Construct a derivation sequence

ϕ_{ij} } from $\mathcal{M}$

$$\left.\begin{array}{l}\psi_0 \\ \psi_1 \\ \vdots\end{array}\right] \text{No assumptions}$$

$\psi_n = \varphi$

Observe (ψ_i) has no assumptions from Σ as all of them are in (ϕ_{ij}). it follows $\mathcal{M} \vdash \varphi$ as (ϕ_{ij}) are from $\mathcal{M}$.

Ex. 4

by def δ satisfies $\mathcal{M}$. Hence $\mathcal{M}$ is consistent by Corollary 2.4.7.
For any $\varphi \in \mathcal{F}^*$, Either

$\delta[\varphi] = 1$	$\delta[\varphi] = 0$
$\varphi \in \Sigma_\delta$ by def. and	$\Sigma_\delta \not\models \varphi$
$\Sigma_\delta \vdash \varphi$ by a one line derv.	By Completeness, $\Sigma_\delta \nvdash \varphi$

Hence either $\Sigma_\delta \vdash \varphi$ or $\Sigma_\delta \nvdash \varphi$.

Ex. 5

1) Yes. Since $\phi \notin \mathcal{M}$ for any formula ϕ on the empty set $\mathcal{M}$, the independency statement is vacuously true. it can be re-written as $\forall \phi (\phi \in \mathcal{M} \to \mathcal{M} \setminus \{\phi\} \not\models \phi)$

2) A singleton is independent iff it's not a tautolgy
Denote the singleton by $S = \{\phi_0\}$
$(\to)$ $S \setminus \{\phi_0\} = \{\} \not\models \phi_0$.
$(\leftarrow)$ $\not\models \phi_0$, which can be re-written as $S \setminus \{\phi_0\} = \{\} \not\models \phi_0$

3) Partial Solution. Assuming finiteness of $\mathcal{M}$

Procedure.
set $\mathcal{M}_0 = \mathcal{M}$
if $\mathcal{M}_i$ is independent, Then we're done
otherwise for a formula ψ_i s.t $\mathcal{M}_i \setminus \{\psi_i\} \models \psi_i$, Construct $\mathcal{M}_{i+1} = \mathcal{M}_i \setminus \{\psi_i\}$

Claim. $\mathcal{M}_{i+1}$ is equivalent to $\mathcal{M}_i$
for an arbitrary truth assignment δ, if δ satisfies $\mathcal{M}_i$, Then trivially satisfies $\mathcal{M}_{i+1}$ also since $\mathcal{M}_{i+1} \subset \mathcal{M}_i$.
if δ satisfies $\mathcal{M}_{i+1}$, then to conclude δ satisfies $\mathcal{M}_i$, it suffices to show δ satisfies ψ_i as well. But we know $\mathcal{M}_i \setminus \{\psi_i\} = \mathcal{M}_{i+1} \models \psi_i$, Concluding δ satisfies $\mathcal{M}_i$.

By finiteness the procedure terminates. Call the final set $\mathcal{M}_K$.
By definition of the procedure this $\mathcal{M}_K$ is independent.

Partial Solution. The statement holds for some infinite $\mathcal{M}$

Construct $\mathcal{M}$ as $\mathcal{M} = \{(P \to P), (P \to (P \to P)), (P \to (P \to (P \to P))), \dots\}$
Clearly all of its elements are tautologies. So $\models \psi$ for any $\psi \in \mathcal{M}$
in other words, there're infinite $\psi \in \mathcal{M}$, s.t $\mathcal{M} \setminus \{\psi\} \models \psi$. Yet the empty set is an independent logically equivalent subset of $\mathcal{M}$.

4) ($\rightarrow$). Trivial by the def.

($\leftarrow$). We show the Contrapositive: dependency of formulas implies the existince of a finite dependent subset of it

Assume a set of formulas M is dependent

Then $\exists \phi \in M$ s.t $M \setminus \{\phi\} \models \phi$

By completeness $M \setminus \{\phi\} \vdash \phi$. Call its derivation sequence (α_i).

Construct $\Sigma = \{ \varphi \in (\alpha_i) \mid \varphi$ is an assumption in $(\alpha_i) \}$

Clearly Σ is finite following from finiteness of (α_i).

Hence $\Sigma \vdash \phi$, and by Soundness $\Sigma \models \phi$.

Construct $\Sigma' = \Sigma \cup \{\phi\}$. Then $\Sigma' \setminus \{\phi\} \models \phi$.

Σ' is a finite dependent subset of M.

5) No. We show any maximal consistent set is dependent.

let M be an arbitrary maximally consistent M.

let $\alpha_0 = (p \to (p \to p))$

Observe

$\vdash \alpha_0$	A1
$M \vdash \alpha_0$	fact 1
$\alpha_0 \in M$	Maximality of M, lemma 2.4.8
$\models \alpha_0$	Soundness and A1
$M \setminus \alpha_0 \models \alpha_0$	fact 2

Concluding M is dependent ∎

Bonus Example.

Consider the empty set. it's consistent by the consistency of system S (thm 2.3.5). Complete by the procedure of thm 2.4.3, and call the resulting set Σ_{com}. This set is maximally consistent (corollary 2.4.11) and dependent by the same line of reasoning.

6) Condition: $\mathcal{M} \setminus \{\phi\}$ is consistent for any $\phi \in \mathcal{M}$ (1)

(1) is Necessary, i.e if an inconsistent set $\mathcal{M}$ is independent then (1) holds.
We show the contrapositive. Assume $\mathcal{M} \setminus \{\phi\}$ is inconsistent for some ϕ. Then it's not satisfiable by theorem 2.3.11. By def for any truth assignment δ, δ doesn't satisfy all formulas of $\mathcal{M} \setminus \{\phi\}$. Then $\mathcal{M} \setminus \{\phi\} \models \phi$ is vacuously true. Hence the inconsistent set $\mathcal{M}$ is dependent.

(1) is Sufficient, i.e if (1) holds, then inconsistent $\mathcal{M}$ is independent
for any formula $\phi \in \mathcal{M}$,

$\mathcal{M} \setminus \{\phi\} \cup \{\phi\} = \mathcal{M}$ is inconsistent — Given

$\mathcal{M} \setminus \{\phi\} \vdash \neg\phi$ — Lemma 2.3.8 & Negation theorem

$\mathcal{M} \setminus \{\phi\} \nvdash \phi$ — Consistency

$\mathcal{M} \setminus \{\phi\} \not\models \phi$ — Completeness