Note:

You are expected to write proofs for the exercises that ask you to compute or to find something

Exercise 1 (10)

Let $\mathcal{L} = \{f\}$ such that f is a binary function symbol. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures such that $\mathcal{M}$ is $(\mathbb{R}, +)$ and $\mathcal{N}$ is $(\mathbb{R}^+, \cdot)$. Show that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.

Exercise 2 (15)

Let $\mathcal{L} = \{E\}$ such that E is a binary relation symbol. Let $\mathcal{G}$ be an $\mathcal{L}$-structure such that $G = \{1, 2, 3, 4\}$, and let $E^{\mathcal{G}} = \{(a, b) \in G^2 \mid a + b \text{ is even}\}$. Give two proper substructures of $\mathcal{G}$ and a non-identity automorphisms of $\mathcal{G}$.

Exercise 3 (20)

Let $\mathcal{L} = \{R\}$ where R is a binary relation symbol. Let $\mathcal{M} = (\mathbb{Z}, <)$ and $\mathcal{N} = (\mathbb{Q}, <)$ be $\mathcal{L}$-structures.
Prove or disprove $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.

Exercise 4 (15)

Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. We call a function $h : M \to N$ an $\mathcal{L}$-homomorphism if and only if:

1. $h(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for all constant symbols c in $\mathcal{L}$.
2. $h(f^{\mathcal{M}}(x_1 \ldots, x_n)) = f^{\mathcal{N}}(h(x_1) \ldots, h(x_n))$ for each n-ary function symbol $f \in \mathcal{L}$ and for all $x_i \in M$.
3. For any relation symbol $R \in \mathcal{L}$ of arity n, if $(x_1 \ldots, x_n) \in \mathcal{R}^{\mathcal{M}}$, then $(h(x_1), \ldots, h(x_n)) \in R^{\mathcal{N}}$.

Give an example of a bijective homomorphism between two structures which is not an isomorphism.

Exercise 5 (Bonus 30)

Let $\mathcal{L} = \{f\} \cup \{f_r | r \in \mathbb{R}\}$ and let $\mathcal{L}' = \mathcal{L} \cup \{g, c\}$. Interpret f to be the binary operation of vector addition, g to be the unary function that sends each vector to its additive inverse, c as the zero vector, and for each f_r interpret it as the unary function of scalar multiplication by r. Formally, let $\mathcal{V}$ be a real vector space, and consider it an $\mathcal{L}'$-structure (or an $\mathcal{L}$-structure if we exclude 1 and 3):

1. $c^{\mathcal{V}} = \underline{\mathbf{0}}$.
2. $f^{\mathcal{V}} : V \times V \to V$ with each $v_1, v_2 \in V$, we have $(v_1, v_2) \mapsto v_1 + v_2$.
3. $g^{\mathcal{V}} : V \to V$ with each $v \in V$, we have $v \mapsto -v$.
4. $f_r^{\mathcal{V}} : V \to V$ with each $v \in V$, we have $v \mapsto rv$ for each $r \in \mathbb{R}$.

Investigate real vector spaces once as $\mathcal{L}$-structures and once as $\mathcal{L}'$-structures. Show whether substructures are subspaces and whether isomorphisms are bijective linear transformations in each language.
Note that we decide on the appropriate language for studying a certain structure based on there two key properties, that is the isomorphism gives you the usual notion of isomorphism (being essentially the same structure) and substructures give the expected notion.

(Hint: Consider the subspace test.**)**

Exercise 6 **(25)**

Suppose that $\mathcal{L}$ has a binary function symbol f, a unary relation symbol P, and a binary relation symbol R. Which of the following strings of symbols are formulas? Present a definition for the notion of a 'subformula' as an extension for the one we did for propositional formulas. For each which is a formula, write down all its subformulas.

1. $\exists x_1(\forall x_3 R(x_2, x_3) \vee (x_1 \to \forall x_1 P(x_1)))$
2. $\forall x_3(P(x_4) \wedge \exists x_1 R(x_1, x_3) \vee P(x_2))$
3. $(\exists x_1 P(x_1) \leftrightarrow \forall x_5 P(f(x_5, x_5)))$

Exercise 7 **(15)**

Suppose that $\mathcal{L}$ has one constant symbol c, a unary function symbol f, a binary function symbol g, a ternary function symbol h, a unary relation symbol P, and a binary relation symbol R. For each of the following formulas, classify all occurrences of variables as free or bound.

1. $\exists x_3(\forall x_2 P(h(x_1, x_3, x_2)) \leftrightarrow (\forall x_1 R(c, x_1) \wedge \forall x_5 P(g(x_3, x_2))))$
2. $(\forall x_5(P(f(x_2)) \vee \exists x_2 R(x_5, x_2)) \to \exists x_3(\forall x_1 R(x_1, x_2) \vee P(x_1)))$