Ex.1

Consider function $h: \mathbb{R} \to \mathbb{R}^+$ where $h(x) = e^x$.

injective. if $h(x) = h(x')$ for $x, x' \in \mathbb{R}$, then

$$e^x = e^{x'}$$
$$\ln e^x = \ln e^{x'}$$
$$x = x'$$

surjective. for any $y \in \mathbb{R}^+$ we know $\ln y = x$ exists.

Then $h(x) = h(\ln y) = e^{\ln y} = y$

preserves interpretation of f. $h(f^M(x_1, x_2))$

$$= h(x_1 + x_2)$$
$$= e^{x_1 + x_2}$$
$$= e^{x_1} \cdot e^{x_2}$$
$$= h(x_1) \cdot h(x_2)$$
$$= f^N(h(x_1), h(x_2))$$

Ex.2

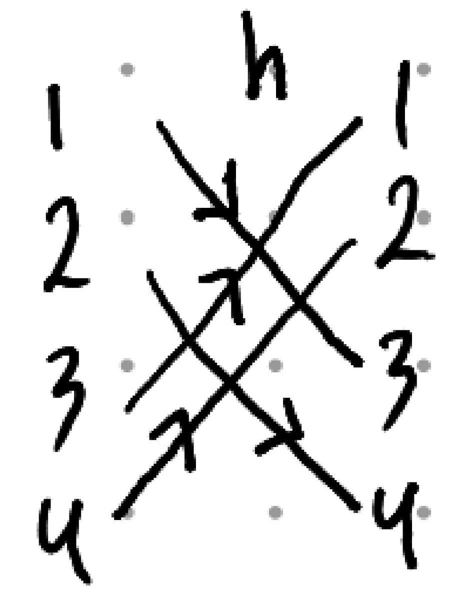
structure $g_0 = (\{\}, \{\})$ is a substructure as it has no members. Observe

$$R^{g_0} = R^g \cap \{\}^2$$
$$\{\} = R^g \cap \{\}$$

structure $g_1 = (\{1\}, \{(1,1)\})$ is a substructure as

$$R^{g_1} = R^g \cap \{1\}^2$$
$$\{(1,1)\} = R^g \cap \{(1,1)\}$$
$$= \{(1,1)\} \quad \text{as } (1,1) \in R^g$$

Construct Automorphism h as



odd
even
odd
even

Observe $h(x)$ is even iff x is even

Observe $(a,b) \in E^{g}$

$\leftrightarrow$ $a+b$ is even

$\leftrightarrow$ Either, a and b are even, or a and b are odd

$\leftrightarrow$ Either, $h(a)$ and $h(b)$ are even, or $h(a)$ and $h(b)$ are odd

$\leftrightarrow$ $h(a)+h(b)$ is even

$\leftrightarrow$ $(h(a), h(b)) \in E^{g}$

Ex. 3

Not Isomorphic. Assume for the sake of contradiction there's a bijection $h: \mathbb{Z} \to \mathbb{Q}$ such that $a<b$ iff $h(a)<h(b)$ for any $a,b \in \mathbb{Z}$. Then $1<2$ implies $h(1)<h(2)$. We know there exists a rational $h(c)$ whereby $h(1)<h(c)<h(2)$. it follow c is an integer where $1<c<2$. Contradiction.

Ex. 4

Consider the two graph structures $\mathcal{M} = (\{x,y,z\}, \{(x,y)\})$ and $\mathcal{N} = (\{a,b,c\}, \{(a,b),(b,c)\})$ with bijection h defined as $x \mapsto a$, $y \mapsto b$, $z \mapsto c$.
Observe $(x,y) \in R^{\mathcal{M}}$ and $(h(x), h(y)) = (a,b) \in R^{\mathcal{N}}$
But $(h(y), h(z)) = (b,c) \in R^{\mathcal{N}}$ and $(y,z) \notin R^{\mathcal{M}}$
Hence homomorphism but not isomorphism.

Ex.6

1) Not a formula

if it were a formula, then by the inductive definition x_1 is a formula also, but it's not.

2) Not a formula

if it were a formula, Then by the inductive definition $P(x_4) \wedge \exists x_1 R(x_1, x_3) \vee P(x_2)$ is also a formula. But it doesn't match any case of the definition. Contradiction.

3) Formula

$$\left(\exists x_1 P(x_1) \leftrightarrow \forall x_5 P(f(x_5, x_5))\right)$$

$$\leftrightarrow$$

$\exists x_1 P(x_1)$ $\qquad\qquad$ $\forall x_5 P(f(x_5, x_5))$

$P(x_1)$ $\qquad\qquad$ $P(f(x_5, x_5))$

The tree's nodes are exactly the subformulas

Subformulas definition Following the same pattern of page 11 definition, For each first-order formula $\theta \in F$, We associate a set $sf[\theta]$, as

* if θ is an atomic formula, Then $sf[\theta] = \{\theta\}$
* if $\theta = \neg \varphi$ for formula φ, Then $sf[\theta] = \{\theta\} \cup sf[\varphi]$
* if $\theta = (\varphi \diamond \psi)$ for formulas φ and ψ, Then $sf[\theta] = sf[\varphi] \cup sf[\psi] \cup \{\theta\}$
* if $\theta = \forall x \varphi$ or $\theta = \exists x \varphi$, Then $sf[\theta] = sf[\varphi] \cup \{\theta\}$

Ex. 7

for both (1) and (2) bound variables x_i are underlined

1) $\exists x_3\ (\forall x_2\, P(h(x_1, \underline{x_3}, \underline{x_2})) \leftrightarrow (\forall x_1\, R(c, \underline{x_1}) \wedge \forall x_5\, P(g(\underline{x_3}, x_2))))$

$$\left(\forall x_2\, P(h(x_1, x_3, x_2)) \leftrightarrow (\forall x_1\, R(c, x_1) \wedge \forall x_5\, P(g(x_3, x_2)))\right)$$

$$\forall x_2\, P(h(x_1, x_3, x_2)) \qquad\qquad \forall x_1\, R(c, x_1) \wedge \forall x_5\, P(g(x_3, x_2))$$

$$P(h(x_1, x_3, x_2)) \qquad\qquad R(c, x_1) \wedge \forall x_5\, P(g(x_3, x_2))$$

$$R(c, x_1) \qquad \forall x_5\, P(g(x_3, x_2))$$

$$P(g(x_3, x_2))$$

x_3 is bound as the formula is of the form $\varphi = \exists x\, \psi$, and by def. (p. 67), none of the occurences is free.

Remaining occurences can be shown by the same line of reasoning.

2) $(\forall x_5\, (P(f(x_2)) \vee \exists x_2\, R(\underline{x_5}, \underline{x_2})) \rightarrow \exists x_3\, (\forall x_1\, R(\underline{x_1}, x_2) \vee P(x_1)))$

Similar reasoning