Note:

You are expected to write proofs for the exercises that ask you to compute or to find something.
You may use any Latin letter (with or without index subscript) as a variable.

Exercise 1 (5+5+5+5)

Let $\mathcal{L} = \{a, b, c, f, R\}$ where a, b, and c are constants symbols, f is a binary function symbol, and R is a binary relation. Consider the structure $\mathcal{M} = \{\mathbb{Q}, 1, 2, 3, +, <\}$. Consider the following formulas:

1. $\forall x\, \exists y\, (f(x, y) = a)$
2. $\forall x\, \neg R(x, a)$
3. $\forall x\, (R(b, a) \to (R(f(x, b), f(x, a))))$
4. $\forall x\, \forall y\ (f(x, y) = z)$

Which of these formulas are satisfied in $\mathcal{M}$? (Show with proof)

Exercise 2 (7+7+8+8)

Let $\mathcal{L} = \{a, b, f, g, R\}$ such that a, b are constant symbols, f, g are binary function symbols, and R is a binary relations symbol. Consider the $\mathcal{L}$-structure $(\mathbb{N}, 0, 1, +, \cdot, <)$. Express each of the following properties about elements in $\mathbb{N}$ with an $\mathcal{L}$-formula.

1. 0 is the smallest natural number.
2. $\varphi(x, y)$ saying x divides y.
3. $\psi(x, y, z)$ saying z is the greatest common divisor of x and y.
4. $\theta(x)$ saying x is a prime number.

Note: For space reasons, you can use a formula from n to write a formula that answers $n + m$ where $m > 0$.

Exercise 3 (40)

Let $\mathcal{L} = \{g\} \cup \{f_r \mid r \in \mathbb{R}\}$ where g is a binary function symbol and each f_r is a unary function symbol. A vector space $\mathcal{V}$ can be viewed as an $\mathcal{L}$-structure by interpreting g as vector addition and f_r is interpreted it as scalar multiplication by r, that is, $f_r^{\mathcal{V}} : V \to V$ given by $f_r^{\mathcal{V}}(v) = rv$ for every vector $v \in V$. Write down axioms of real vector spaces in the first-order language $\mathcal{L}$ (i.e. provide a set of $\mathcal{L}$-sentences Γ in which an $\mathcal{L}$-structure $\mathcal{M} \models \Gamma$ if and only if M is a real vector space with the operations given by the interpretations in $\mathcal{M}$).

Exercise 4 (10)

If $\Gamma \models \forall x\, \varphi(x)$ then $\Gamma \models \varphi(\tau)$, for any term τ. Prove or disprove this statement.
Hint: consider $\varphi(x) = \exists y\, \neg(x = y)$