Ex. 1

1) $\mathcal{M} \models \forall x \exists y \, (f(x,y) = a)$
iff for each member $b \in Q$, $\mathcal{M} \models \psi(b/x)$
iff for each member $b \in Q$,
there's at least one element $b_1 \in Q$, $\mathcal{M} \models \psi_1(b, b_1)$
iff for each member $b \in Q$, there's $b_1 \in Q$, such that
$(+(b, b_1), 1) \in <$, or in other notation
$b + b_1 < 1$

$$\begin{aligned} \varphi &= \forall x \exists y \, (f(x,y) = a) \\ &= \forall x \, \psi(x) \\ &= \forall x \exists y \, (\psi_1(x,y)) \\ &= \forall x \exists y \, (=(f(x,y), a)) \\ &= \forall x \exists y \, (\psi_2(x,y)) \end{aligned}$$

it holds by the additive inverse existince of rational numbers.

2) $\mathcal{M} \models \forall x \, \neg R(x, a)$
iff for $b \in Q$, $\mathcal{M} \models \psi(b/x)$
iff for $b \in Q$, $\mathcal{M} \not\models \psi_1(b/x)$
iff for $b \in Q$, $(b, 1) \notin <$, or in another notation
$b \not< 1$

$$\begin{aligned} \varphi &= \forall x \, \psi(x) \\ &= \forall x \neg \psi_1(x) \end{aligned}$$

But $1/2 \in Q$ and $1/2 < 1$. it follows $\mathcal{M} \not\models \forall x \, \neg R(x, a)$

3) $\mathcal{M} \models \forall x \left(R(b, a) \rightarrow (R(f(x,b), f(x,a))) \right)$
iff for $b \in Q$, $\mathcal{M} \models \psi(b/x)$
iff for $b \in Q$, Either $\mathcal{M} \not\models \psi_1(b/x)$ or $\mathcal{M} \models \psi_2(b/x)$

$$\begin{aligned} \varphi &= \forall x \, \psi(x) \\ &= \forall x \, \psi_1(x) \rightarrow \psi_2(x) \end{aligned}$$

but $\mathcal{M} \not\models \psi_1(b/x)$ is equivalent to $2 \not< 1$ which holds.

Then φ is satisfied in $\mathcal{M}$

4) Call $\varphi(z) = \forall x \forall y\, (f(x,y) = z)$

We show it's not satisfiable on any member $r \in Q$, Concluding $\mathcal{M} \not\models \varphi$

$$\varphi(1/z) = \forall x \forall y\, (f(x,y) = z) = \forall x\, \psi(x, 1/z) = \forall x \forall y\, \psi_1(x, y, \tfrac{1}{z})$$

$\mathcal{M} \models \varphi(r/z)$

iff for $b \in Q$, $\mathcal{M} \models \psi(b/x, r/z)$

iff for $b, b_1 \in Q$, $\mathcal{M} \models \psi_1(b/x, b_1/y, r/z)$

iff for $b, b_1 \in Q$, $=^{\mathcal{M}}(f^{\mathcal{M}}(\frac{b}{x}, \frac{b_1}{y}), z^{\mathcal{M}})$, or in another notation $b + b_1 = r$

But it doesn't hold. for example $r/4 \in Q$ and $r/4 + r/4 \neq r$

Hence Not satisfiable.

Ex. 2

Note. The convention for the first-order languages is to contain equality symbol "="

1) $\varphi = \forall x \big(R(0,x) \vee =(0,x)\big)$

$\varphi^N = \forall x\big(R^N(0,x) \vee =^N(0,x)\big)$

$= \forall x \big(0 < x \vee 0 = x\big)$

2) $\varphi(x,y) = \exists k =(g(x,k), y)$

$\varphi^N(x,y) = \exists k =^N(g^N(x,k), y)$

$= \exists k\ x \cdot k = y$

3) $\psi(x,y,z) = \Big(\big(\varphi(z,x) \wedge \varphi(z,y)\big) \wedge \Big(\forall k\big(\varphi(k,x) \wedge \varphi(k,y)\big) \to \big(=(k,z) \vee R(k,z)\big)\Big)\Big)$

$\psi^N(x,y,z)$

$= \Big(\big(\varphi^N(z,x) \wedge \varphi^N(z,y)\big) \wedge \Big(\forall k\big(\varphi^N(k,x) \wedge \varphi^N(k,y)\big) \to \big(=^N(k,z) \vee R^N(k,z)\big)\Big)\Big)$

$= \Big(\big(\exists k\ z \cdot k = x \wedge \exists k\ z \cdot k = y\big) \wedge \big(\forall k\,(\exists z\ k \cdot z = x \wedge \exists z\ k \cdot z = y) \to k = z \vee k < z\big)\Big)$

4) $\theta(x) = \Big(\forall k\ \varphi(k,x) \to \big(=(k,x) \vee =(k,b)\big)\Big)$

$\theta^N(x) = \Big(\forall k\ \varphi^N(k,x) \to \big(=^N(k,x) \vee =^N(k,b^N)\big)\Big)$

$= \big(\forall k\ \exists z\ k \cdot z = x \to (k = x \vee k = 1)\big)$

Ex. 3

Vector addition associativity

$\forall u \forall v \forall w = \left(g(u, g(v, w)), g(g(u,v), w) \right)$ $\quad$ $u+(v+w) = (u+v)+w$

Vector addition Commutativity

$\forall u \forall v = (g(u,v), g(v,u))$ $\quad$ $u+v = v+u$

Vector addition identity

$\exists v \, \forall u \; g(f_0^V(v), u) = g(u, f_0^V(v)) = u$ $\quad$ $\exists v \in V \; \forall u \in V$ $\; 0+u = u+0 = u$

alt. $\exists r \in R \; \exists v \, \forall u \; g(f_r^V(v), u) = g(u, f_r^V(v)) = u$

Vector addition inverse

$\forall v \, \exists u \; g(v, u) = f_0^V(u)$ $\quad$ $\forall v \in V \; \exists u \in V \; v+u=0$

Scalar multiplication Compatibility

for any a and b in R, $\forall v \; f_a(f_b^V(v)) = f_{ab}^V(v)$ $\quad$ $\forall v \in V$, $a, b \in F$ $\; a(b \cdot v) = (ab) \cdot v$

Scalar multiplication identity

$\forall v \; f_1^V(v) = v$ $\quad$ $\forall v \in V$ $\; 1 \cdot v = v$

Scalar multiplication distributivity

for any a in R, $\forall v \forall u \; f_a^V(g(u,v)) = g(f_a^V(u), f_a^V(v))$ $\quad$ $\forall u, v \in V$, $\forall a \in F$ $\; a(u+v) = au + av$

for any a and b in R, $\forall v \; f_{a+b}^V(v) = g(f_a^V(v), f_b^V(v))$ $\quad$ $\forall v \in V$, $\forall a, b \in F$ $\; (a+b)v = av + bv$

The interpretation of these formulas under a structure is satisfied if and only if the structure satisfies real vector spaces axioms.

For example, Consider $\mathcal{V} = (V, 0, \oplus, \otimes)$, satisfying for any $v \in V$.

* $0 \oplus v = v \oplus 0 = v$ (1)
* $0 \otimes v = 0$ (2)

For ℓ-formula $\varphi = \exists v \, \forall u \; g(f_0^{\mathcal{V}}(v), u) = g(u, f_0^{\mathcal{V}}(v))$, Observe

$\mathcal{V} \models \varphi$

iff

there's a member $b \in V$, such that for all $b_1 \in V$,

$$0 \otimes b + b_1 = b_1 + 0 \otimes b$$
$$0 + b_1 = b_1 + 0 = b_1$$

Ex. 4

We disprove by a counter-example.

Consider the structure $\mathcal{N} = (N, I)$ where N is the set of natural numbers and I is a unary identity function $I: x \mapsto x$.

let as hinted $\varphi(x) = \exists y \neg (x = y)$.

Clearly $\mathcal{N} \models \forall x\, \varphi(x)$, i.e for every natural number there's a distinct natural number from it.

Consider the term $T = g(y)$, with function symbol g and variable symbol y, and The formula $\varphi(T) = \exists y \neg (T = y) = \exists y \neg (g(y) = y)$.

Whereby $g^{\mathcal{N}}$ is the identity I, $\varphi(T)$ is not satisfied under structure $\mathcal{N}$. By def the interpretation means there's a natural number b such that $I(y) \neq y$, Contradiction.

In Conclusion, $\mathcal{N} \not\models \varphi(T)$ for the term T.

We are given for any member $b \in \Gamma$, $\Gamma \models \varphi(b/x)$

Consider an arbitrary term T.

Case 1 T is a constant

Then T^{Γ} is a member of Γ.

Hence $M \models \varphi(T^{M}/x)$

Case 2 T is a variable x_i

by hypothesis we know $M \models (b/x)$ for any $b \in \Gamma$ flaw Possibly variables

Case 3 $T = f(t_1, \ldots, t_n)$ for some terms $t_1, \ldots, t_n$

Then $T^{M} = f^{M}(t_1^{M}, \ldots, t_n^{M}) \in \Gamma$, A member of Γ

Hence $M \models \varphi(T^{M}/x)$ Not well-defined

From Case 1, Case 2, and Case 3, $M \models \varphi(T)$ for any term T.