Ex. 1

lemma. For any formulas α, β, γ

1. $\alpha \vdash (\beta \to \alpha)$

2. $\{(\gamma \to \alpha), (\alpha \to \beta)\} \vdash (\gamma \to \beta)$

3. $\{(\alpha \to (\beta \to \gamma)), \beta\} \vdash (\alpha \to \gamma)$

4. $\vdash (x = x)$

5. $\vdash (x = y \to y = x)$

proof.

1.
α Assumption
$(\alpha \to (\beta \to \alpha))$ Ax. 1
$(\beta \to \alpha)$ MP

2.
$((\gamma \to (\alpha \to \beta)) \to ((\gamma \to \alpha) \to (\gamma \to \beta)))$ Ax.2
$(\alpha \to \beta)$ Assumption
$(\gamma \to (\alpha \to \beta))$ lemma 1
$((\gamma \to \alpha) \to (\gamma \to \beta))$ MP
$(\gamma \to \alpha)$ Assumption
$(\gamma \to \beta)$ MP

3. $\{\beta,\ \alpha \to (\beta \to \gamma)\} \vdash (\alpha \to \gamma)$

$(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$ Ax. 2

β Assumption

$(\alpha \to \beta)$ lemma 1

$(\alpha \to (\beta \to \gamma))$ Assumption

$((\alpha \to \beta) \to (\alpha \to \gamma))$ MP

$(\alpha \to \gamma)$ MP

4.
Given $\vdash (t = t)$ in p. 84, it immediately follows $\vdash (x = x)$

5. $\vdash (x = y \to y = x)$

$x = y \to ((x = x) \to (y = x))$ Ax. 7

$(x = x)$ lemma 4

$(x = y \to y = x)$ lemma 3

1)

$(y = x \to (y = z \to x = z))$ Ax. 7

$(x = y \to y = x)$ lemma 5

$(x = y \to (y = z \to x = z))$ lemma 2

2)

Claim. $\{\forall x \forall y\, \varphi\} \vdash \forall y \forall x\, \varphi$

$\forall x \forall y\, \varphi$	Assump.
$\forall x \forall y\, \varphi \rightarrow \forall y\, \varphi(x)$	Ax. 4
$\forall y\, \varphi(x)$	MP
$\forall y\, \varphi(x) \rightarrow \varphi(x,y)$	Ax. 4
$\varphi(x,y)$	MP
$\forall x\, \varphi(y)$	Gen
$\forall y \forall x\, \varphi$	Gen

Since all occurences of x and y in $\forall x \forall y\, \varphi$ are bound, we can use the deduction theorem to conclude

$\vdash (\forall x \forall y\, \varphi \rightarrow \forall y \forall x\, \varphi)$ ∎

Ex. 2

$\mathcal{M} \models \left(\forall x(\varphi \to \psi) \to (\varphi \to \forall x \psi) \right)$, where x isn't free in φ

iff if $\mathcal{M} \models \forall x(\varphi \to \psi)$, and if $\mathcal{M} \models \varphi$, Then $\mathcal{M} \models \forall x \psi$

iff if

* for any $b \in M$, if $\mathcal{M} \models \varphi$, then $\mathcal{M} \models \psi(b/x)$. Observe we substituted $\varphi(x)$ by φ as x is not free in φ. They're the same formulas.
* $\mathcal{M} \models \varphi$

Then $\mathcal{M} \models \psi(b/x)$ for any $b \in M$

The last statement clearly holds. ■

Ex. 3

This is exactly the contrapositive of thm 4.4.5

Ex. 4

1) We prove it by induction along the pairing axiom.
Claim. For all $n \geq 1$, Given $X_1, X_2, \cdots, X_n$ are sets, So is $\bigcup_{i=1}^{n} X_i$.

Base Case. $n=1$. $\bigcup_{i=1}^{1} X_i = X_1$, already given as a set.

Induction Step. For $n>1$, We know both $\bigcup_{i=1}^{n-1} X_i$ and X_n are sets.

Then $\bigcup_{i=1}^{n} X_i = \bigcup_{i=1}^{n-1} X_i \cup X_n$ is a set by pairing axiom.

2) Already given in lemma 5.1.1.